Metric Lie algebras and quadratic extensions

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Abstract

The present paper contains a systematic study of the structure of metric Lie algebras, i.e., finite-dimensional real Lie algebras equipped with a non-degenerate invariant symmetric bilinear form. We show that any metric Lie algebra $\mathfrak{g}$ without simple ideals has the structure of a so called balanced quadratic extension of an auxiliary Lie algebra $\mathfrak{l}$ by an orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ in a canonical way. Identifying equivalence classes of quadratic extensions of $\mathfrak{l}$ by $\mathfrak{a}$ with a certain cohomology set $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ we obtain a classification scheme for general metric Lie algebras and a complete classification of metric Lie algebras of index 3.

Contents

1 Introduction

The present paper is an attempt towards the classification of metric Lie algebras up to isomorphism. Here a metric Lie algebra is a finite-dimensional real Lie algebra equipped with an invariant non-degenerate symmetric bilinear form. An isomorphism of metric Lie algebras is by definition a Lie algebra isomorphism which is in addition an isometry with respect to the given inner products.

A metric Lie algebra $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ is called decomposable if it contains a proper ideal $\mathfrak{k}$ which is non-degenerate (i.e. $\langle\cdot\,,\cdot\rangle_{|\mathfrak{k}\times\mathfrak{k}}$ is non-degenerate), and indecomposable otherwise. Then any metric Lie algebra is the orthogonal direct sum of uniquely determined indecomposable metric Lie algebras. Note that indecomposable metric Lie algebras are either simple or do not contain simple ideals. Therefore, one is lead to investigate indecomposable metric Lie algebras without simple ideals.

Our study is mainly motivated by the wish of understanding general pseudo-Riemannian symmetric spaces. Indeed, the Lie algebra of the transvection group of a pseudo-Riemannian symmetric space has the structure of a metric Lie algebra ([CP 80], Prop. 1.6). Moreover, the local geometry of a pseudo-Riemannian symmetric space is completely determined by this metric Lie algebra together with the isometric involutive automorphism of it induced by the geodesic symmetry. Our theory of metric Lie algebras is designed in such a way that the incorporation of an involutive automorphism into the structure does not present serious additional difficulties. Thus there is a theory of pseudo-Riemannian symmetric spaces completely parallel to the theory of metric Lie algebras developed in the present paper. The details will appear in a forthcoming paper.

For further motivation and remarks on the history of the subject we refer to the introduction of [KO 02]. In the paper [KO 02] we studied the relatively special class of metric Lie algebras $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ which satisfy $\mathfrak{g}'' \subset \mathfrak{z}(\mathfrak{g})$. One consequence of this investigation was the complete classification of indecomposable metric Lie algebras of index 2 (this classification has been already announced in [BK 02], the classification of metric Lie algebras of index 1 is due to Medina [M 85]).

In the present paper we are able to carry over the approach of [KO 02] to general metric Lie algebras. Let us outline the main ideas and results of the paper.

The basic construction, which goes back to an idea of Berard Bergery used in his unpublished work on pseudo-Riemannian symmetric spaces [BB2], is the following: Let $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ be a metric Lie algebra. Assume that we have an isotropic ideal $\mathfrak{i} \subset \mathfrak{g}$ such that $\mathfrak{i}^\perp/\mathfrak{i}$ is abelian. Here $\mathfrak{i}^\perp$ denotes the orthogonal "complement" of $\mathfrak{i}$. Set $\mathfrak{l} = \mathfrak{g}/\mathfrak{i}^\perp$ and $\mathfrak{a} = \mathfrak{i}^\perp/\mathfrak{i}$. Then $\mathfrak{a}$ inherits an inner product from $\mathfrak{g}$ and an $\mathfrak{l}$-action respecting this inner product, i.e., it inherits the structure of an orthogonal $\mathfrak{l}$-module. Moreover, $\mathfrak{i} \cong \mathfrak{l}^*$ as an $\mathfrak{l}$-module, and $\mathfrak{g}$ can be represented as the result of two subsequent extensions of Lie algebras with abelian kernel

$$0 \to \mathfrak{a} \longrightarrow \mathfrak{g}/\mathfrak{i} \longrightarrow \mathfrak{l} \to 0\ , \qquad 0 \to \mathfrak{l}^* \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{i} \to 0\ . \tag{1}$$

Vice versa, given a Lie algebra $\mathfrak{l}$, an orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ and two extensions as in (1) which in addition satisfy certain natural compatibility conditions, the resulting Lie algebra $\mathfrak{g}$ has a distinguished invariant inner product. This construction of metric Lie algebras, being a relative of the double extension method of Medina and Revoy [MR 85], will be formalised into the notion of a quadratic extension of $\mathfrak{l}$ by an orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ in Subsection 3.1. In particular, there is a natural equivalence relation on the set of quadratic extensions of $\mathfrak{l}$ by $\mathfrak{a}$. The cocycles defining the extensions in (1) represent an element in a certain cohomology set $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$, and it turns out that there is a bijection between equivalence classes of such quadratic extensions and $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$

(Theorem 3.1). We introduce these cohomology sets and study their basic functorial properties in Section 2. Cohomology sets of this kind were studied first by Grishkov [Gr 98].

What makes the theory of quadratic extensions so useful is the fact that any metric Lie algebra $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ without simple ideals has a canonical isotropic ideal $\mathfrak{i} = \mathfrak{i}(\mathfrak{g})$ (Definitions 4.2 and 4.3) such that $\mathfrak{i}(\mathfrak{g})^\perp/\mathfrak{i}(\mathfrak{g})$ is abelian. In other words, $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ has a canonical structure of a quadratic extension of $\mathfrak{l} = \mathfrak{g}/\mathfrak{i}(\mathfrak{g})^\perp$ by $\mathfrak{a} = \mathfrak{i}(\mathfrak{g})^\perp/\mathfrak{i}(\mathfrak{g})$ (Proposition 4.1). It has the property that the orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ is semi-simple. However, not every quadratic extension of a Lie algebra $\mathfrak{l}$ by a semi-simple orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ arises in this way. The obvious condition that the image of $\mathfrak{l}^*$ in $\mathfrak{g}$ in (1) should be equal to $\mathfrak{i}(\mathfrak{g})$ is not always satisfied. If it is satisfied we call the quadratic extension balanced and the corresponding cohomology class in $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ admissible. The main result of Section 4 is Theorem 4.1 which characterises the admissible cohomology classes in $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$.

It is now easy to decide which admissible cohomology classes correspond to decomposable metric Lie algebras (see Section 5, in particular Definition 5.1 for the definition of indecomposable cohomology classes). We denote the set of indecomposable admissible cohomology classes by $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$. Moreover, it turns out that elements of $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$ correspond to isomorphic Lie algebras if and only if they can be transformed into each other by the induced action of the automorphisms group $G_{\mathfrak{l},\mathfrak{a}}$ of the pair $(\mathfrak{l},\mathfrak{a})$. Thus we can identify the set of isomorphism classes of non-simple indecomposable metric Lie algebras with the union of orbit spaces

$$\coprod_{\mathfrak{l},\mathfrak{a}} \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0/G_{\mathfrak{l},\mathfrak{a}}\ , \tag{2}$$

where the union is taken over a set of representatives of isomorphism classes of pairs $(\mathfrak{l},\mathfrak{a})$ consisting of real finite-dimensional Lie algebras $\mathfrak{l}$ and a semi-simple orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ (Theorem 5.1). This is the classification scheme we aimed at. In particular, it says that an arbitrary non-simple indecomposable Lie algebra can be constructed as the quadratic extension corresponding to an element of $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$ for some pair $(\mathfrak{l},\mathfrak{a})$ and it clearly indicates when two metric Lie algebras constructed in this way are isomorphic.

In order to approach a true classification one has to evaluate (2) further. One first observes (see Section 6) that the class of Lie algebras $\mathfrak{l}$ which really occurs in (2), i.e., $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 \neq \emptyset$ for some orthogonal $\mathfrak{l}$-module $\mathfrak{a}$, is a proper subclass of all Lie algebras. The critical condition is that there has to exist an admissible cohomology class for some orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ (in contrast, the indecomposability condition is harmless and gives no restrictions on the Lie algebra $\mathfrak{l}$). It is an open question how large this class of Lie algebras really is; see the comment at the end of Section 5. Some partial results in this direction are obtained in Section 6.

By construction, the dimension of the Lie algebra $\mathfrak{l}$ associated with a metric Lie algebra $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ is bounded by the index of $\langle\cdot\,,\cdot\rangle$. For this reason the bijection (2) is extremely useful for a concrete classification of metric Lie algebras with small index, where all the ingredients of (2) can be explicitely computed. This is demonstrated in Section 7 for index 3.

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2 Quadratic cohomology

Let $\mathfrak{l}$ be a finite-dimensional real Lie algebra, and let $\rho : \mathfrak{l} \to \mathfrak{gl}(\mathfrak{a})$ be a representation of $\mathfrak{l}$ on a finite-dimensional real vector space $\mathfrak{a}$ equipped with an invariant symmetric bilinear form $\langle\cdot\,,\cdot\rangle$, i.e. for $L \in \mathfrak{l}$ and $v, w \in \mathfrak{a}$

$$\langle \rho(L)v, w\rangle + \langle v, \rho(L)w\rangle = 0 \ .$$

In addition, we require $\langle\cdot\,,\cdot\rangle$ to be non-degenerate. Then the triple $(\rho, \mathfrak{a}, \langle\cdot\,,\cdot\rangle)$ is called an orthogonal $\mathfrak{l}$-module. Often ρ or $\langle\cdot\,,\cdot\rangle$ will be omitted in the notation.

The goal of this section is to associate with an orthogonal $\mathfrak{l}$-module a sequence of sets

$$\mathcal{H}^p_Q(\mathfrak{l}, \mathfrak{a}) \ , \quad p \in 2\mathbb{N}_0 \ ,$$

the quadratic cohomology sets of $\mathfrak{l}$ with coefficients in $(\rho, \mathfrak{a}, \langle\cdot\,,\cdot\rangle)$. Though they are specializations of the cohomology sets associated by Grishkov [Gr 98] to a cochain complex with a cup product taking values in a second cochain complex we prefer to present a self-contained treatment here.

Let us recall the construction of usual Lie algebra cohomology. For any representation $\rho : \mathfrak{l} \to \mathfrak{gl}(\mathfrak{a})$ of a Lie algebra $\mathfrak{l}$ on a vector space $\mathfrak{a}$ we have the standard Lie algebra cochain complex $(C^*(\mathfrak{l}, \mathfrak{a}), d)$, where $C^p(\mathfrak{l}, \mathfrak{a}) = \mathrm{Hom}(\bigwedge^p \mathfrak{l}, \mathfrak{a})$, and for $\tau \in C^p(\mathfrak{l}, \mathfrak{a})$

$$\begin{aligned} d\tau(L_1, \ldots, L_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i-1} \rho(L_i)\tau(L_1, \ldots, \hat{L}_i, \ldots, L_{p+1}) \\ &\quad + \sum_{i<j} (-1)^{i+j} \tau([L_i, L_j], L_1, \ldots, \hat{L}_i, \ldots, \hat{L}_j, \ldots, L_{p+1}) \ . \end{aligned}$$

We denote the groups of cocycles and coboundaries of $C^p(\mathfrak{l}, \mathfrak{a})$ by $Z^p(\mathfrak{l}, \mathfrak{a})$ and $B^p(\mathfrak{l}, \mathfrak{a})$, respectively. Then the quotients

$$H^p(\mathfrak{l}, \mathfrak{a}) := Z^p(\mathfrak{l}, \mathfrak{a})/B^p(\mathfrak{l}, \mathfrak{a})$$

constitute the cohomology groups of $\mathfrak{l}$ with coefficients in $\mathfrak{a}$.

Now let $(\rho, \mathfrak{a}, \langle\cdot\,,\cdot\rangle)$ be an orthogonal $\mathfrak{l}$-module. By $C^*(\mathfrak{l})$ we denote the cochain complex associated to the trivial one-dimensional representation. Then the composition of maps

$$C^p(\mathfrak{l}, \mathfrak{a}) \times C^q(\mathfrak{l}, \mathfrak{a}) \xrightarrow{\wedge} C^{p+q}(\mathfrak{l}, \mathfrak{a} \otimes \mathfrak{a}) \xrightarrow{\langle\cdot\,,\cdot\rangle} C^{p+q}(\mathfrak{l})$$

defines a bilinear multiplication

$$C^p(\mathfrak{l}, \mathfrak{a}) \times C^q(\mathfrak{l}, \mathfrak{a}) \to C^{p+q}(\mathfrak{l})$$

which we will denote by $(\alpha,\tau)\mapsto\langle\alpha\wedge\tau\rangle$. In concrete terms

$$\begin{aligned}&\langle\alpha\wedge\tau\rangle(L_1,\dots,L_{p+q})\\ &\quad=\sum_{[\sigma]\in\mathfrak{S}_{p+q}/\mathfrak{S}_p\times\mathfrak{S}_q}\operatorname{sgn}(\sigma)\langle\alpha(L_{\sigma(1)},\dots,L_{\sigma(p)}),\tau(L_{\sigma(p+1)},\dots,L_{\sigma(p+q)})\rangle\ ,\end{aligned}$$

where $\mathfrak{S}_k$ denotes the symmetric group in k letters. As a consequence of the invariance of $\langle\cdot\,,\cdot\rangle$ we have for $\alpha\in C^p(\mathfrak{l},\mathfrak{a})$ and $\tau\in C^q(\mathfrak{l},\mathfrak{a})$

$$d\langle\alpha\wedge\tau\rangle=\langle d\alpha\wedge\tau\rangle+(-1)^p\langle\alpha\wedge d\tau\rangle\ . \tag{3}$$

It follows that $\langle\cdot\wedge\cdot\rangle$ induces a kind of cup product on the cohomology groups

$$\cup: H^p(\mathfrak{l},\mathfrak{a})\times H^q(\mathfrak{l},\mathfrak{a})\longrightarrow H^{p+q}(\mathfrak{l})\ ,\quad [\alpha]\cup[\tau]:=[\langle\alpha\wedge\tau\rangle]\ . \tag{4}$$

Set $n=\dim\mathfrak{l}$. If $\mathfrak{l}$ is unimodular which is equivalent to $H^n(\mathfrak{l})\neq\{0\}$, then we obtain a non-degenerate pairing (Poincaré duality)

$$\cup: H^p(\mathfrak{l},\mathfrak{a})\times H^{n-p}(\mathfrak{l},\mathfrak{a})\longrightarrow H^n(\mathfrak{l})\cong\mathbb{R}. \tag{5}$$

Definition 2.1 *Let p be even. We define a group structure on $\mathcal{C}_Q^{p-1}(\mathfrak{l},\mathfrak{a}):=C^{p-1}(\mathfrak{l},\mathfrak{a})\oplus C^{2p-2}(\mathfrak{l})$ by*

$$(\tau_1,\sigma_1)*(\tau_2,\sigma_2):=(\tau_1+\tau_2,\sigma_1+\sigma_2+\tfrac12\langle\tau_1\wedge\tau_2\rangle)\ .$$

We call $\mathcal{C}_Q^{p-1}(\mathfrak{l},\mathfrak{a})$ the group of quadratic $(p-1)$-cochains. The set of quadratic p-cocycles is given by

$$\mathcal{Z}_Q^p(\mathfrak{l},\mathfrak{a}):=\{(\alpha,\gamma)\in C^p(\mathfrak{l},\mathfrak{a})\oplus C^{2p-1}(\mathfrak{l})\mid d\alpha=0,\ d\gamma=\tfrac12\langle\alpha\wedge\alpha\rangle\}\ .$$

Clearly, $*$ is associative. The inverse of (τ,σ) is given by $(-\tau,\frac12\langle\tau\wedge\tau\rangle-\sigma)$. Thus $*$ indeed defines a group structure on $\mathcal{C}_Q^{p-1}(\mathfrak{l},\mathfrak{a})$.

Lemma 2.1 *Let p be even. Let $(\alpha,\gamma)\in C^p(\mathfrak{l},\mathfrak{a})\oplus C^{2p-1}(\mathfrak{l})$ and $(\tau,\sigma)\in C^{p-1}(\mathfrak{l},\mathfrak{a})\oplus C^{2p-2}(\mathfrak{l})$. Then the formula*

$$(\alpha,\gamma)(\tau,\sigma):=\Big(\alpha+d\tau,\gamma+d\sigma+\langle(\alpha+\tfrac12 d\tau)\wedge\tau\rangle\Big) \tag{6}$$

defines a right action of the group $\mathcal{C}_Q^{p-1}(\mathfrak{l},\mathfrak{a})$ on $C^p(\mathfrak{l},\mathfrak{a})\oplus C^{2p-1}(\mathfrak{l})$. This action leaves the set of p-cocycles $\mathcal{Z}_Q^p(\mathfrak{l},\mathfrak{a})\subset C^p(\mathfrak{l},\mathfrak{a})\oplus C^{2p-1}(\mathfrak{l})$ invariant.

Proof. Let $(\tau_i,\sigma_i)\in C^{p-1}(\mathfrak{l},\mathfrak{a})\oplus C^{2p-2}(\mathfrak{l})$, $i=1,2$. Since p is even we have by (3) that

$$d\langle\tau_1\wedge\tau_2\rangle=\langle d\tau_1\wedge\tau_2\rangle-\langle d\tau_2\wedge\tau_1\rangle\ .$$

We obtain

$$\tfrac12\left(\langle d\langle\tau_1\wedge\tau_2\rangle+\langle d\tau_1\wedge\tau_2\rangle+\langle d\tau_2\wedge\tau_1\rangle\right)=\langle d\tau_1\wedge\tau_2\rangle\ .$$

Therefore we have for $(\alpha,\gamma)\in C^p(\mathfrak{l},\mathfrak{a})\oplus C^{2p-1}(\mathfrak{l})$

$$\begin{aligned}
&(\alpha,\gamma)((\tau_1,\sigma_1)*(\tau_2,\sigma_2))\\
&\quad= (\alpha,\gamma)+\Big(d(\tau_1+\tau_2), d(\sigma_1+\sigma_2+\tfrac12\langle\tau_1\wedge\tau_2\rangle)+\langle(\alpha+\tfrac12 d(\tau_1+\tau_2))\wedge(\tau_1+\tau_2)\rangle\Big)\\
&\quad= \Big(\alpha+d\tau_1,\gamma+d\sigma_1+\langle(\alpha+\tfrac12 d\tau_1)\wedge\tau_1\rangle\Big)+\Big(d\tau_2, d\sigma_2+\langle(\alpha+d\tau_1+\tfrac12 d\tau_2)\wedge\tau_2\rangle\Big)\\
&\quad= ((\alpha,\gamma)(\tau_1,\sigma_1))(\tau_2,\sigma_2)\ .
\end{aligned}$$

This proves the first assertion of the lemma.

Now let $(\alpha,\gamma)\in\mathcal{Z}^p_Q(\mathfrak{l},\mathfrak{a})$, i.e. $d\alpha=0$, $d\gamma=\frac12\langle\alpha\wedge\alpha\rangle$. Using again Equation (3) we find for $(\tau,\sigma)\in\mathcal{C}^{p-1}_Q(\mathfrak{l},\mathfrak{a})$

$$\begin{aligned}
d(\gamma+d\sigma+\langle(\alpha+\tfrac12 d\tau)\wedge\tau\rangle) &= \tfrac12\langle\alpha\wedge\alpha\rangle+\langle(\alpha+\tfrac12 d\tau)\wedge d\tau\rangle\\
&= \tfrac12\langle(\alpha+d\tau)\wedge(\alpha+d\tau)\rangle\ .
\end{aligned}$$

This implies $(\alpha,\gamma)(\tau,\sigma)\in\mathcal{Z}^p_Q(\mathfrak{l},\mathfrak{a})$. The proof of the lemma is now complete. □

Remark 2.1 The subgroup $Z^{2p-2}(\mathfrak{l})\subset\mathcal{C}^{p-1}(\mathfrak{l},\mathfrak{a})$ acts trivially on $C^p(\mathfrak{l},\mathfrak{a})\oplus C^{2p-1}(\mathfrak{l})$. Thus the above action amounts to an action of the group

$$\mathcal{B}^p_Q(\mathfrak{l},\mathfrak{a}) := (C^{p-1}(\mathfrak{l},\mathfrak{a})\oplus B^{2p-1}(\mathfrak{l}),*)\ ,$$

where the multiplication $*$ is given by

$$(\tau_1,\sigma_1)*(\tau_2,\sigma_2) := (\tau_1+\tau_2,\sigma_1+\sigma_2+\tfrac12 d\langle\tau_1\wedge\tau_2\rangle)\ .$$

Definition 2.2 *For any $p\in 2\mathbb{N}_0$ we define the p-th cohomology set $\mathcal{H}^p_Q(\mathfrak{l},\mathfrak{a})$ of $\mathfrak{l}$ with coefficients in $(\mathfrak{a},\langle\cdot,\cdot\rangle)$ as the quotient space of the action of $\mathcal{C}^{p-1}_Q(\mathfrak{l},\mathfrak{a})$ (or $\mathcal{B}^p_Q(\mathfrak{l},\mathfrak{a})$) on $\mathcal{Z}^p_Q(\mathfrak{l},\mathfrak{a})$. If $(\alpha,\gamma)\in\mathcal{Z}^p_Q(\mathfrak{l},\mathfrak{a})$, then we denote the corresponding cohomology class in $\mathcal{H}^p_Q(\mathfrak{l},\mathfrak{a})$ by $[\alpha,\gamma]$.*

The sets $\mathcal{H}^p_Q(\mathfrak{l},\mathfrak{a})$ appear among the cohomology sets introduced and studied by Grishkov in [Gr 98]. There they are denoted by H^{2p-1}_Δ, where Δ is the multiplication given by $\frac12\langle\cdot\wedge\cdot\rangle$. Grishkov also defines corresponding cohomology sets for odd p in an analogous way. We have avoided to discuss them here because they do not give anything new. In fact, they turn out to be canonically isomorphic to $H^p(\mathfrak{l},\mathfrak{a})\oplus H^{2p-1}(\mathfrak{l})$.

Note that $\mathcal{H}^0_Q(\mathfrak{l},\mathfrak{a})$ is equal to the set of isotropic invariant vectors in $\mathfrak{a}$. In the present paper the set $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ will be the main technical tool for studying metric Lie algebras. For abelian $\mathfrak{l}$ this set has been already intensively investigated in [KO 02].

We will need the basic functorial properties of the cohomology sets $\mathcal{H}^p_Q(\mathfrak{l},\mathfrak{a})$, $p\in 2\mathbb{N}_0$. Similar as usual Lie algebra cohomology groups they turn out to be contravariant with respect to Lie algebras and covariant with respect to modules.

We consider the category $\mathcal{LO}$ of pairs $(\mathfrak{l}, \mathfrak{a})$ of Lie algebras $\mathfrak{l}$ and orthogonal $\mathfrak{l}$-modules. Morphisms $(\mathfrak{l}_1, \mathfrak{a}_1) \mapsto (\mathfrak{l}_2, \mathfrak{a}_2)$ are given by pairs (S, U), where $S : \mathfrak{l}_1 \to \mathfrak{l}_2$ is a Lie algebra homomorphism and $U : \mathfrak{a}_2 \to \mathfrak{a}_1$ is an isometric embedding such that

$$U \circ \rho_2(S(L)) = \rho_1(L) \circ U \ . \tag{7}$$

Notice that U maps in the reverse direction. The condition (7) means that $U \in \mathrm{Hom}_{\mathfrak{l}_1}(\mathfrak{a}_2, \mathfrak{a}_1)$, where the $\mathfrak{l}_1$-module structure of $\mathfrak{a}_2$ is given by $S^*\rho_2 := \rho_2 \circ S$. We call morphisms in $\mathcal{LO}$ morphisms of pairs.

Let $(S, U) : (\mathfrak{l}_1, \mathfrak{a}_1) \to (\mathfrak{l}_2, \mathfrak{a}_2)$ be a morphism of pairs. For all $p \in \mathbb{N}_0$ we then have the pull back maps

$$S^* : C^p(\mathfrak{l}_2) \longrightarrow C^p(\mathfrak{l}_1) \ , \quad S^*\gamma(L_1, \ldots, L_p) := \gamma(S(L_1), \ldots, S(L_p))$$

and

$$(S, U)^* : C^p(\mathfrak{l}_2, \mathfrak{a}_2) \longrightarrow C^p(\mathfrak{l}_1, \mathfrak{a}_1) \ , \quad (S, U)^*\alpha(L_1, \ldots, L_p) := U \circ \alpha(S(L_1), \ldots, S(L_p)) \ .$$

S^* and $(S, U)^*$ commute with the differentials. Moreover we have

$$\langle (S, U)^*\alpha \wedge (S, U)^*\tau \rangle = S^*\langle \alpha \wedge \tau \rangle \ .$$

For this property it is crucial that U is an isometric embedding. Thus the direct sum $(S, U)^* \oplus S^*$ maps the space of cocycles $\mathcal{Z}^p_Q(\mathfrak{l}_2, \mathfrak{a}_2) \subset C^p(\mathfrak{l}_2, \mathfrak{a}_2) \oplus C^{2p-1}(\mathfrak{l}_2)$ to $\mathcal{Z}^p_Q(\mathfrak{l}_1, \mathfrak{a}_1)$ and defines a group homomorphism from $\mathcal{C}^{p-1}_Q(\mathfrak{l}_2, \mathfrak{a}_2)$ to $\mathcal{C}^{p-1}_Q(\mathfrak{l}_1, \mathfrak{a}_1)$. Moreover, $(S, U)^* \oplus S^*$ intertwines the $\mathcal{C}^{p-1}_Q$-actions defined in Lemma 2.1:

$$((S, U)^* \oplus S^*)((\alpha, \gamma)(\tau, \sigma)) = ((S, U)^*\alpha, S^*\gamma)((S, U)^*\tau, S^*\sigma) \ .$$

We obtain

Proposition 2.1 *Let $F = (S, U) : (\mathfrak{l}_1, \mathfrak{a}_1) \to (\mathfrak{l}_2, \mathfrak{a}_2)$ be a morphism of pairs. Then for $p \in 2\mathbb{N}_0$ there is a pull back map*

$$F^* : \mathcal{H}^p_Q(\mathfrak{l}_2, \mathfrak{a}_2) \longrightarrow \mathcal{H}^p_Q(\mathfrak{l}_1, \mathfrak{a}_1)$$

given by

$$F^*[\alpha, \gamma] := [(S, U)^*\alpha, S^*\gamma] \ , \quad (\alpha, \gamma) \in \mathcal{Z}^p_Q(\mathfrak{l}_2, \mathfrak{a}_2) \ .$$

Therefore the assignment $(\mathfrak{l}, \mathfrak{a}) \rightsquigarrow \mathcal{H}^p_Q(\mathfrak{l}, \mathfrak{a})$ is a contravariant functor from the category $\mathcal{LO}$ to the category of sets.

For any orthogonal $\mathfrak{l}$-module $(\rho, \mathfrak{a})$ and any $L \in \mathfrak{l}$ the element $I_L = (e^{\mathrm{ad}(L)}, e^{-\rho(L)})$ is an automorphism of the pair $(\mathfrak{l}, \mathfrak{a})$. By definition, the group of inner automorphisms of $(\mathfrak{l}, \mathfrak{a})$ is the group generated by the elements I_L, $L \in \mathfrak{l}$. It is well known that inner automorphisms act trivially on usual Lie algebra cohomology. We now show that the same is true for quadratic cohomology.

Proposition 2.2 *Let $(\rho, \mathfrak{a})$ be an orthogonal $\mathfrak{l}$-module. Then for any $L \in \mathfrak{l}$ and $p \in 2\mathbb{N}_0$*

$$I_L^* = \mathrm{Id}_{\mathcal{H}_Q^p(\mathfrak{l},\mathfrak{a})} \ .$$

Proof. We look at the one parameter group

$$t \mapsto \Phi_t = ((e^{-\operatorname{ad}(tL)}, e^{\rho(tL)})^*, (e^{-\operatorname{ad}(tL)})^*)$$

acting on $\mathcal{C}_Q^p(\mathfrak{l},\mathfrak{a})$. Let X_L be the corresponding vector field on $\mathcal{C}_Q^p(\mathfrak{l},\mathfrak{a})$, $X_L(\alpha,\gamma) := \frac{d}{dt}|_{t=0}\Phi_t(\alpha,\gamma)$. Inserting L in the first place of a cochain defines maps $i_L : C^q(\mathfrak{l},\mathfrak{a}) \to C^{q-1}(\mathfrak{l},\mathfrak{a})$. By the well known homotopy formula for the Lie derivative we have

$$X_L(\alpha,\gamma) = ((d \circ i_L + i_L \circ d)\alpha, (d \circ i_L + i_L \circ d)\gamma) \ .$$

Set $\tau_L = i_L(\alpha)$, $\sigma_L = i_L(\gamma)$. If $(\alpha,\gamma) \in \mathcal{Z}_Q^p(\mathfrak{l},\mathfrak{a})$, then

$$X_L(\alpha,\gamma) = (d\tau_L, d\sigma_L + \tfrac{1}{2} i_L\langle\alpha \wedge \alpha\rangle) = (d\tau_L, d\sigma_L + \langle\tau_L \wedge \alpha\rangle) \ .$$

This formula shows that X_L is tangential to $\mathcal{C}_Q^{p-1}(\mathfrak{l},\mathfrak{a})$-orbits in $\mathcal{Z}_Q^p(\mathfrak{l},\mathfrak{a})$. Therefore, Φ_t maps each orbit to itself. In other words, the induced action of Φ_t on cohomology is trivial. Since I_L^* is induced by Φ_{-1} the proposition follows. $\square$

Definition 2.3 *The direct sum of two pairs $(\mathfrak{l}_i, \mathfrak{a}_i)$, $i = 1, 2$, is defined by*

$$(\mathfrak{l},\mathfrak{a}) = (\mathfrak{l}_1,\mathfrak{a}_1) \oplus (\mathfrak{l}_2,\mathfrak{a}_2) := (\mathfrak{l}_1 \oplus \mathfrak{l}_2, \mathfrak{a}_1 \oplus \mathfrak{a}_2) \ ,$$

where the direct sum of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ is orthogonal, and for $i \neq j$ the Lie algebra $\mathfrak{l}_i$ acts trivially on $\mathfrak{a}_j$. A direct sum is called non-trivial if both summands are different from the pair $(0,0)$.

The following lemma can be easily verified.

Lemma 2.2 *Let $(\mathfrak{l},\mathfrak{a}) = (\mathfrak{l}_1,\mathfrak{a}_1) \oplus (\mathfrak{l}_2,\mathfrak{a}_2)$ be the direct sum of two pairs. Let $q_i : \mathfrak{l} \to \mathfrak{l}_i$ be the projection, and let $j_i : \mathfrak{a}_i \to \mathfrak{a}$ be the injection. Then there is a natural injective map*

$$+ \ : \ \Big((q_1,j_1)^*\mathcal{H}_Q^p(\mathfrak{l}_1,\mathfrak{a}_1)\Big) \times \Big((q_2,j_2)^*\mathcal{H}_Q^p(\mathfrak{l}_2,\mathfrak{a}_2)\Big) \longrightarrow \mathcal{H}_Q^p(\mathfrak{l},\mathfrak{a})$$

induced by addition in $\mathcal{Z}_Q^p(\mathfrak{l},\mathfrak{a}) \subset Z^p(\mathfrak{l},\mathfrak{a}) \oplus C^{2p-1}(\mathfrak{l})$.

We conclude this section by clarifying the relationship between $\mathcal{H}_Q^p(\mathfrak{l},\mathfrak{a})$ and the cohomology groups $H^*(\mathfrak{l},\mathfrak{a})$ and $H^*(\mathfrak{l})$. Recall the definition (4) of the cup product $\cup$. We consider the subvariety $H_\cup^p(\mathfrak{l},\mathfrak{a}) \subset H^p(\mathfrak{l},\mathfrak{a})$ defined by

$$H_\cup^p(\mathfrak{l},\mathfrak{a}) := \{a \in H^p(\mathfrak{l},\mathfrak{a}) \mid a \cup a = 0\} \ .$$

The map $\tilde{p} : \mathcal{Z}_Q^p(\mathfrak{l},\mathfrak{a}) \to H^p(\mathfrak{l},\mathfrak{a})$, $(\alpha,\gamma) \mapsto [\alpha]$, is constant along $\mathcal{C}_Q^{p-1}(\mathfrak{l},\mathfrak{a})$-orbits and has image $H_\cup^p(\mathfrak{l},\mathfrak{a})$. Thus it induces a surjective map $p : \mathcal{H}_Q^p(\mathfrak{l},\mathfrak{a}) \to H_\cup^p(\mathfrak{l},\mathfrak{a})$.

Proposition 2.3 *The map* $p : \mathcal{H}^p_Q(\mathfrak{l},\mathfrak{a}) \to H^p_\cup(\mathfrak{l},\mathfrak{a})$ *gives rise to a partition of* $\mathcal{H}^p_Q(\mathfrak{l},\mathfrak{a})$

$$\mathcal{H}^p_Q(\mathfrak{l},\mathfrak{a}) = \coprod_{a\in H^p_\cup(\mathfrak{l},\mathfrak{a})} p^{-1}(a)$$

into affine spaces $p^{-1}(a)$ *with associated vector spaces* $H^{2p-1}(\mathfrak{l})/(a\cup H^{p-1}(\mathfrak{l},\mathfrak{a}))$. *The affine structure of* $p^{-1}(a)$ *is given by the formula*

$$p^{-1}(a) \times H^{2p-1}(\mathfrak{l})/(a\cup H^{p-1}(\mathfrak{l},\mathfrak{a})) \ni ([\alpha,\gamma],[\delta]) \longmapsto [\alpha,\gamma+\delta] \in p^{-1}(a)\ . \qquad (8)$$

Here $(\alpha,\gamma)\in\tilde{p}^{-1}(a)$ *and* $\delta\in Z^{2p-1}(\mathfrak{l})$.

Proof. We have to check that the action (8) is well-defined and simply transitive. First we observe that the abelian group $Z^{2p-1}(\mathfrak{l})$ acts on $\tilde{p}^{-1}(a)$ by

$$(\alpha,\gamma)\delta \mapsto (\alpha,\gamma+\delta)\ . \qquad (9)$$

This action commutes with the action of $\mathcal{C}^{p-1}_Q(\mathfrak{l},\mathfrak{a})$. The resulting action of $\mathcal{C}^{p-1}_Q(\mathfrak{l},\mathfrak{a})\times Z^{2p-1}(\mathfrak{l})$ on $\tilde{p}^{-1}(a)$ is transitive. Therefore we obtain a well-defined transitive action of $Z^{2p-1}(\mathfrak{l})$ on $p^{-1}(a)$. We compute its kernel. We have

$$[\alpha,\gamma+\delta]=[\alpha,\gamma] \ \Leftrightarrow\ \exists\,\tau\in Z^{p-1}(\mathfrak{l},\mathfrak{a}),\sigma\in C^{2p-2}(\mathfrak{l}) \text{ s.th. } \delta=\langle\alpha\wedge\tau\rangle+d\sigma\ .$$

Thus (9) induces a simply transitive action of

$$Z^{2p-1}(\mathfrak{l})/(\langle\alpha\wedge Z^{p-1}(\mathfrak{l},\mathfrak{a})\rangle + B^{2p-1}(\mathfrak{l})) \cong H^{2p-1}(\mathfrak{l})/(a\cup H^{p-1}(\mathfrak{l},\mathfrak{a}))$$

on $p^{-1}(a)$ which coincides with (8). □

3 Quadratic extensions

In this section we study a two step extension procedure, called quadratic extension, of Lie algebras by orthogonal modules resulting in metric Lie algebras. For a fixed Lie algebra $\mathfrak{l}$ and an orthogonal module $\mathfrak{a}$ we establish a bijection between equivalence classes of quadratic extensions of $\mathfrak{l}$ by $\mathfrak{a}$ and the cohomology set $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$. In particular, for any $(\alpha,\gamma)\in\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$ we construct a metric Lie algebra which has the structure of a quadratic extension.

3.1 Definition

Let $\mathfrak{l}$ be a Lie algebra and $(\rho,\mathfrak{a},\langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ an orthogonal $\mathfrak{l}$-module. We will also consider $\mathfrak{a}$ as an abelian metric Lie algebra.

Definition 3.1 *A quadratic extension of* $\mathfrak{l}$ *by* $\mathfrak{a}$ *is given by a quadrupel* $(\mathfrak{g},\mathfrak{i},i,p)$*, where*

- $\mathfrak{g}$ *is a metric Lie algebra*

- $\mathfrak{i} \subset \mathfrak{g}$ *is an isotropic ideal and*
- i *and* p *are Lie algebra homomorphisms constituting an exact sequence of Lie algebras*
$$0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{g}/\mathfrak{i} \xrightarrow{p} \mathfrak{l} \to 0 \ , \tag{10}$$
which is consistent with the representation ρ *of* $\mathfrak{l}$ *on* $\mathfrak{a}$ *in the following sense:*
$$i(\rho(L)A) = [\tilde{L}, i(A)] \ \in \ i(A) \tag{11}$$
holds for all $\tilde{L} \in \mathfrak{g}/\mathfrak{i}$ *with* $p(\tilde{L}) = L$. *In addition we require that* $\operatorname{im} i = \mathfrak{i}^\perp/\mathfrak{i}$ *and that* $i : \mathfrak{a} \to \mathfrak{i}^\perp/\mathfrak{i}$ *is an isometry.*

Recall that an isotropic ideal of a metric Lie algebra is always abelian, hence $\mathfrak{i} \subset \mathfrak{g}$ is abelian.

If $\mathfrak{g}$ is a metric Lie algebra with an isotropic ideal $\mathfrak{i} \in \mathfrak{g}$ such that $\mathfrak{i}^\perp/\mathfrak{i}$ is abelian, then the sequence
$$0 \to \mathfrak{i}^\perp/\mathfrak{i} \xrightarrow{i} \mathfrak{g}/\mathfrak{i} \xrightarrow{p} \mathfrak{g}/\mathfrak{i}^\perp \to 0 \tag{12}$$
defines a quadratic extension of $\mathfrak{g}/\mathfrak{i}^\perp$ by the orthogonal module $\mathfrak{i}^\perp/\mathfrak{i}$. We call (12) the canonical extension associated with $(\mathfrak{g}, \mathfrak{i})$.

Let $\tilde{p} : \mathfrak{g} \to \mathfrak{l}$ be the composition of the natural projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ with p. Now let $p^* := \mathfrak{l}^* \to \mathfrak{g}$ be the dual map of $\tilde{p}$, where we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using the non-degenerate inner product on $\mathfrak{g}$. This homomorphism is injective since $\tilde{p}$ is surjective. Its image equals $(\ker \tilde{p})^\perp = \mathfrak{i}$.

Using the homomorphism p^* we see that a quadratic extension determines a second exact sequence of Lie algebras
$$0 \to \mathfrak{l}^* \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{i} \to 0 \ , \tag{13}$$
where we consider $\mathfrak{l}^*$ as abelian Lie algebra.

Equations (10) and (13) show that a metric Lie algebra which is a quadratic extension of $\mathfrak{l}$ by $\mathfrak{a}$ can be considered as the result of two subsequent extensions of Lie algebras which satisfy certain compatibility conditions: first an extension of $\mathfrak{l}$ by $\mathfrak{a}$ and second an extension of the resulting Lie algebra by $\mathfrak{l}^*$. This is the point of view taken in [KO 02], where the equivalent notion of a twofold extension was studied (for abelian $\mathfrak{l}$).

Lemma 3.1 *Let* $(\mathfrak{g}, \mathfrak{i}, i, p)$ *be a quadratic extension of* $\mathfrak{l}$ *by* $\mathfrak{a}$. *Then* $\mathfrak{g}$ *does not contain a simple ideal.*

Proof. Assume that $\mathfrak{s}$ is a simple ideal in $\mathfrak{g}$. Then $\mathfrak{s}$ does not contain non-zero abelian ideals. Hence, $\mathfrak{i} \cap \mathfrak{s} = 0$. In particular, $[\mathfrak{i}, \mathfrak{s}] = 0$. This implies
$$\langle \mathfrak{i}, \mathfrak{s} \rangle = \langle \mathfrak{i}, [\mathfrak{s}, \mathfrak{s}] \rangle = \langle [\mathfrak{i}, \mathfrak{s}], \mathfrak{s} \rangle = 0,$$
thus $\mathfrak{s} \subset \mathfrak{i}^\perp$. We conclude $\mathfrak{s} \subset \mathfrak{i}^\perp/\mathfrak{i}$, which contradicts the assumption that $\mathfrak{a}$ is abelian. $\square$

Definition 3.2 *Two quadratic extensions $(\mathfrak{g}_j, \mathfrak{i}_j, i_j, p_j)$, $j = 1, 2$, of $\mathfrak{l}$ by $\mathfrak{a}$ are called to be equivalent if there exists an isomorphism of metric Lie algebras $\Psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ which maps $\mathfrak{i}_1$ onto $\mathfrak{i}_2$ and satisfies*

$$\overline{\Psi} \circ i_1 = i_2 \qquad \text{and} \qquad p_2 \circ \overline{\Psi} = p_1 \ ,$$

where $\overline{\Psi} : \mathfrak{g}_1/\mathfrak{i}_1 \to \mathfrak{g}_2/\mathfrak{i}_2$ is the induced map.

3.2 The standard model

Let $\mathfrak{l}$ be a Lie algebra and let $(\rho, \mathfrak{a}, \langle\cdot\,,\cdot\rangle_\mathfrak{a})$ be an orthogonal $\mathfrak{l}$-module. We choose $\alpha \in C^2(\mathfrak{l}, \mathfrak{a})$ and $\gamma \in C^3(\mathfrak{l})$. We consider the vector space $\mathfrak{d} := \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}$ and define an inner product $\langle\cdot\,,\cdot\rangle$ on $\mathfrak{d}$ by

$$\langle Z_1 + A_1 + L_1, Z_2 + A_2 + L_2\rangle = \langle A_1, A_2\rangle_\mathfrak{a} + Z_1(L_2) + Z_2(L_1)$$

for all $Z_1, Z_2 \in \mathfrak{l}^*$, $A_1, A_2 \in \mathfrak{a}$ and $L_1, L_2 \in \mathfrak{l}$.

Proposition 3.1 *(i) There exists a unique antisymmetric bilinear map $[\cdot\,,\cdot] : \mathfrak{d}\times\mathfrak{d} \to \mathfrak{d}$ which satisfies*

$$\langle\cdot\,,\cdot\rangle \text{ is invariant, i.e. } \langle[X, Z], Y\rangle = \langle X, [Z, Y]\rangle \text{ for all } X, Y, Z \in \mathfrak{d}, \quad (14)$$

$$[\mathfrak{d}, \mathfrak{l}^*] \subset \mathfrak{l}^*, \ \ [\mathfrak{l}^*, \mathfrak{l}^*] = 0, \ \ [\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{l}^*, \quad (15)$$

$$[L_1, L_2] = \gamma(L_1, L_2, \cdot) + \alpha(L_1, L_2) + [L_1, L_2]_\mathfrak{l} \quad \text{for all } L_1, L_2 \in \mathfrak{l}, \quad (16)$$

$$\langle[L, A_1], A_2\rangle = \langle\rho(L)A_1, A_2\rangle \quad \text{for all } L \in \mathfrak{l},\ A_1, A_2 \in \mathfrak{a}. \quad (17)$$

(ii) The triple $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}, \rho) := (\,\mathfrak{d}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle\,)$ is a metric Lie algebra if and only if $(\alpha, \gamma) \in \mathcal{Z}^2_Q(\mathfrak{l}, \mathfrak{a})$.

Proof. Assume $[\cdot\,,\cdot]$ is an antisymmetric bilinear map satisfying Equations (14) to (17). By $[\mathfrak{l}^*, \mathfrak{d}] \subset \mathfrak{l}^*$ and the invariance of $\langle\cdot\,,\cdot\rangle$ we have $[(\mathfrak{l}^*)^\perp, \mathfrak{d}] \subset (\mathfrak{l}^*)^\perp$, thus $[\mathfrak{l}^*+\mathfrak{a}, \mathfrak{d}] \subset \mathfrak{l}^*+\mathfrak{a}$. Hence, $[L, A]$ is in $\mathfrak{l}^* + \mathfrak{a}$ for all $L \in \mathfrak{l}$ and $A \in \mathfrak{a}$. By the invariance of $\langle\cdot\,,\cdot\rangle$ and Equation (16) we obtain

$$\langle[L, A], L'\rangle = -\langle A, [L, L']\rangle = -\langle A, \alpha(L, L')\rangle$$

for all $L' \in \mathfrak{l}$. Together with Equation (17) this yields

$$[L, A] = \rho(L)A - \langle A, \alpha(L, \cdot)\rangle. \quad (18)$$

Similarly, we obtain

$$[L, Z] = \mathrm{ad}^*(L)Z \quad (19)$$

$$[A_1, A_2] = \langle\rho(\cdot)A_1, A_2\rangle \quad (20)$$

$$[A, \sigma] = 0 \quad (21)$$

for all $L \in \mathfrak{l}$, $Z \in \mathfrak{l}^*$ and $A, A_1, A_2 \in \mathfrak{a}$. Hence, if $[\cdot\,,\cdot]$ is an antisymmetric bilinear map satisfying (14) to (17), then it is uniquely determined. On the other hand we can

define an antisymmetric bilinear map by Equations (16) to (21) and $[\mathfrak{l}^*,\mathfrak{l}^*]=0$. Then $\langle\cdot\,,\cdot\rangle$ is invariant and Equation (15) holds. This proves the first assertion of the lemma.

The triple $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$ is a metric Lie algebra if and only if $[\cdot\,,\cdot]$ as defined in *(i)* satisfies the Jacobi identity. Obviously, for any $\alpha\in C^2(\mathfrak{l},\mathfrak{a})$ and $\gamma\in C^3(\mathfrak{l})$ the map $[\cdot\,,\cdot]$ defined above satisfies

$$[\mathfrak{l},[\mathfrak{l}^*+\mathfrak{a},\mathfrak{l}^*]]=[\mathfrak{l}^*+\mathfrak{a},[\mathfrak{l}^*,\mathfrak{l}]]=[\mathfrak{l}^*,[\mathfrak{l},\mathfrak{l}^*+\mathfrak{a}]]=0$$

and

$$[\mathfrak{l}^*+\mathfrak{a},[\mathfrak{l}^*+\mathfrak{a},\mathfrak{l}^*+\mathfrak{a}]]=0.$$

Furthermore, it satisfies

$$\begin{aligned}&[L_1,[L_2,Z]]+[L_2,[Z,L_1]]+[Z,[L_1,L_2]]\\&\qquad=\operatorname{ad}^*(L_1)\operatorname{ad}^*(L_2)Z-\operatorname{ad}^*(L_2)\operatorname{ad}^*(L_1)Z-\operatorname{ad}^*([L_1,L_2])Z=0\end{aligned}$$

and

$$\begin{aligned}&[A_1,[A_2,L]]+[A_2,[L,A_1]]+[L,[A_1,A_2]]\\&\qquad=\langle\rho(\cdot)A_1,-\rho(L)A_2\rangle+\langle\rho(\cdot)A_2,\rho(L)A_1\rangle-\langle\rho([L,\cdot\,])A_1,A_2\rangle=0\end{aligned}$$

since ρ is an orthogonal representation. We have to prove that the remaining identities

$$\sum_{\text{cycl}}[L_1,[L_2,L_3]]=0 \tag{22}$$

$$[A,[L_1,L_2]]+[L_1,[L_2,A]]+[L_2,[A,L_1]]=0 \tag{23}$$

for $L_1,L_2,L_3\in\mathfrak{l}$ and $A\in\mathfrak{a}$ are equivalent to the condition $(\alpha,\gamma)\in\mathcal{Z}^p_Q$. Here $\sum_{\text{cycl}}$ denotes the sum over all cyclic permutations of L_1,L_2 and L_3. Because of

$$\begin{aligned}\sum_{\text{cycl}}[L_1,[L_2,L_3]]&=\sum_{\text{cycl}}[L_1,\gamma(L_2,L_3,\cdot\,)+\alpha(L_2,L_3)+[L_2,L_3]_{\mathfrak{l}}]\\&=\sum_{\text{cycl}}\Big(-\gamma(L_2,L_3,[L_1,\cdot\,])+\rho(L_1)\alpha(L_2,L_3)-\langle\alpha(L_2,L_3),\alpha(L_1,\cdot\,)\rangle\\&\qquad\qquad+\gamma(L_1,[L_2,L_3]_{\mathfrak{l}},\cdot\,)+\alpha(L_1,[L_2,L_3]_{\mathfrak{l}})+[L_1,[L_2,L_3]_{\mathfrak{l}}]_{\mathfrak{l}}\Big)\\&=(d\gamma-\frac{1}{2}\langle\alpha\wedge\alpha\rangle)(L_1,L_2,L_3,\cdot\,)+d\alpha(L_1,L_2,L_3)\ \in\mathfrak{l}^*\oplus\mathfrak{a}\end{aligned}$$

Equation (22) is equivalent with $d\gamma=\frac{1}{2}\langle\alpha\wedge\alpha\rangle$ and $d\alpha=0$. Similarly one proves

$$[A,[L_1,L_2]]+[L_1,[L_2,A]]+[L_2,[A,L_1]]=\langle A,-d\alpha(L_1,L_2,\cdot\,)\rangle.$$

Hence, Equation (23) is equivalent to $d\alpha=0$. This proves the second assertion of the lemma. $\square$

We identify $\mathfrak{d}/\mathfrak{l}^*$ with $\mathfrak{a}\oplus\mathfrak{l}$ and denote by $i:\mathfrak{a}\to\mathfrak{a}\oplus\mathfrak{l}$ the injection and by $p:\mathfrak{a}\oplus\mathfrak{l}\to\mathfrak{l}$ the projection. Then the following proposition is obvious.

Proposition 3.2 *If $(\alpha,\gamma)\in\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$, then the quadrupel $(\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho),\mathfrak{l}^*,i,p)$ is a quadratic extension of $\mathfrak{l}$ by $\mathfrak{a}$.*

We will denote the quadratic extension $(\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho),\mathfrak{l}^*,i,p)$ also by $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$.

Remark 3.1 Let $(\alpha,\gamma)\in\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$ be a cocycle and $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)=(\,\mathfrak{d},[\cdot\,,\cdot],\langle\cdot\,,\cdot\rangle\,)$ the associated metric Lie algebra constructed above. Furthermore, let $\langle\cdot\,,\cdot\rangle_{\mathfrak{l}}$ be an invariant not necessarily non-degenerate inner product on $\mathfrak{l}$. We define a new scalar product $\langle\cdot\,,\cdot\rangle'$ on $\mathfrak{d}$ by $\langle\cdot\,,\cdot\rangle'=\langle\cdot\,,\cdot\rangle\oplus\langle\cdot\,,\cdot\rangle_{\mathfrak{l}}$. Then $\mathfrak{d}'_{\alpha,\gamma}(\mathfrak{l},\langle\cdot\,,\cdot\rangle_{\mathfrak{l}},\mathfrak{a},\rho)=(\,\mathfrak{d},[\cdot\,,\cdot],\langle\cdot\,,\cdot\rangle'\,)$ is also a metric Lie algebra. Moreover, $(\mathfrak{d}'_{\alpha,\gamma}(\mathfrak{l},\langle\cdot\,,\cdot\rangle_{\mathfrak{l}},\mathfrak{a},\rho),\mathfrak{l}^*,i,p)$ with i and p as above is a quadratic extension of $\mathfrak{l}$ by $\mathfrak{a}$ and

$$\begin{array}{rcl}\mathfrak{d}'_{\alpha,\gamma}(\mathfrak{l},\langle\cdot\,,\cdot\rangle_{\mathfrak{l}},\mathfrak{a},\rho) & \longrightarrow & \mathfrak{d}_{\alpha,\gamma-\frac{1}{2}\langle[\cdot\,,\cdot]_{\mathfrak{l}},\cdot\,\rangle}(\mathfrak{l},\mathfrak{a},\rho)\\ Z+A+L & \longmapsto & Z+A+L+\dfrac{1}{2}\langle L,\cdot\,\rangle_{\mathfrak{l}}\end{array}$$

for $Z\in\mathfrak{l}^*$, $A\in\mathfrak{a}$, $L\in\mathfrak{l}$ is an equivalence of quadratic extensions.

3.3 Classification by cohomology

Proposition 3.3 *For $(\alpha_1,\gamma_1),\ (\alpha_2,\gamma_2)\in\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$ the quadratic extensions $\mathfrak{d}_{\alpha_1,\gamma_1}(\mathfrak{l},\mathfrak{a},\rho)$ and $\mathfrak{d}_{\alpha_2,\gamma_2}(\mathfrak{l},\mathfrak{a},\rho)$ of $\mathfrak{l}$ by $\mathfrak{a}$ are equivalent if and only if $[\alpha_1,\gamma_1]=[\alpha_2,\gamma_2]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$.*

Proof. A linear map $\Psi:\mathfrak{d}_{\alpha_1,\gamma_1}(\mathfrak{l},\mathfrak{a},\rho)\to\mathfrak{d}_{\alpha_2,\gamma_2}(\mathfrak{l},\mathfrak{a},\rho)$ is an equivalence of quadratic extensions if and only if

(i) $\Psi(\mathfrak{l}^*)=\mathfrak{l}^*$, $\mathrm{proj}_{\mathfrak{a}}\Psi|_{\mathfrak{a}}=\mathrm{Id}$, $\mathrm{proj}_{\mathfrak{l}}\Psi|_{\mathfrak{l}}=\mathrm{Id}$,

(ii) Ψ is an isometry, and

(iii) Ψ is a Lie algebra isomorphism.

Condition (i) is equivalent to

$$\Psi=\begin{pmatrix}\psi & \eta & \xi\\ 0 & \mathrm{Id} & \tau\\ 0 & 0 & \mathrm{Id}\end{pmatrix}:\mathfrak{l}^*\oplus\mathfrak{a}\oplus\mathfrak{l}\longrightarrow\mathfrak{l}^*\oplus\mathfrak{a}\oplus\mathfrak{l}\tag{24}$$

for linear maps $\psi:\mathfrak{l}^*\to\mathfrak{l}^*,\ \eta:\mathfrak{a}\to\mathfrak{l}^*,\ \xi:\mathfrak{l}\to\mathfrak{l}^*,\ \tau:\mathfrak{l}\to\mathfrak{a}$.

Conditions (i) and (ii) are satisfied if and only if Ψ is as above and the equations

$$\begin{array}{rclcl}\langle Z,L\rangle &=& \langle\Psi Z,\Psi L\rangle &=& \langle\psi Z,\xi L+\tau L+L\rangle=\langle\psi Z,L\rangle\\ 0 &=& \langle\Psi A,\Psi L\rangle &=& \langle\eta A+A,\xi L+\tau L+L\rangle\ =\langle\eta A,L\rangle+\langle A,\tau L\rangle\\ 0 &=& \langle\Psi L_1,\Psi L_2\rangle &=& \langle\xi L_1+\tau L_1+L_1,\xi L_2+\tau L_2+L_2\rangle\\ &&&=& \langle\xi L_1,L_2\rangle+\langle L_1,\xi L_2\rangle+\langle\tau L_1,\tau L_2\rangle.\end{array}$$

hold. Let $\xi^*:\mathfrak{l}\to\mathfrak{l}^*$ and $\tau^*:\mathfrak{a}\to\mathfrak{l}^*$ be the dual maps of ξ and τ, respectively. These maps are given by $\langle\xi^*L_1,L_2\rangle=\langle L_1,\xi L_2\rangle$ and $\langle\tau^*A,L\rangle=\langle A,\tau L\rangle$ for $A\in\mathfrak{a}$, $L,L_1,L_2\in\mathfrak{l}$. Then the last equation says that the selfdual part $\frac{1}{2}(\xi+\xi^*)$ of ξ equals $-\frac{1}{2}\tau^*\tau$.

Consequently, Conditions (i) and (ii) are satisfied if and only if Ψ is as in (24) with $\psi = \mathrm{Id}$, $\eta = -\tau^*$ and $\xi = \bar\sigma - \frac{1}{2}\tau^*\tau$ for an anti-selfdual map $\bar\sigma : \mathfrak{l} \to \mathfrak{l}^*$, i.e. if and only if

$$\Psi = \Psi(\tau, \sigma) := \begin{pmatrix} \mathrm{Id} & -\tau^* & \bar\sigma - \frac{1}{2}\tau^*\tau \\ 0 & \mathrm{Id} & \tau \\ 0 & 0 & \mathrm{Id} \end{pmatrix} : \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l} \longrightarrow \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}, \tag{25}$$

where $\tau \in C^1(\mathfrak{l}, \mathfrak{a})$ and $\sigma(\cdot\,,\cdot) = \langle \bar\sigma(\cdot), \cdot\rangle \in C^2(\mathfrak{l})$.

Now we consider Condition (iii). We denote the Lie brackets on $\mathfrak{d}_{\alpha_1,\gamma_1}(\mathfrak{l}, \mathfrak{a}, \rho)$ and $\mathfrak{d}_{\alpha_2,\gamma_2}(\mathfrak{l}, \mathfrak{a}, \rho)$ by $[\cdot\,,\cdot]_1$ and $[\cdot\,,\cdot]_2$, respectively. Assume Ψ is given as in (25). Then Ψ is a Lie algebra isomorphism if and only if the Lie bracket $[\cdot\,,\cdot]'$ defined by $[X, Y]' := \Psi^{-1}[\Psi X, \Psi Y]_2$ for all $X, Y \in \mathfrak{d}$ is equal to $[\cdot\,,\cdot]_1$. By Proposition 3.1 this is the case if and only if $[\cdot\,,\cdot]'$ satisfies

$$\begin{aligned}
&\langle [X, Z]', Y\rangle = \langle X, [Z, Y]'\rangle \text{ for all } X, Y, Z \in \mathfrak{d},\\
&[\mathfrak{l}^*, \mathfrak{d}]' \subset \mathfrak{l}^*,\ \ [\mathfrak{l}^*, \mathfrak{l}^*]' = 0,\ \ [\mathfrak{a}, \mathfrak{a}]' \subset \mathfrak{l}^*,\\
&[L_1, L_2]' = \gamma_1(L_1, L_2, \cdot) + \alpha_1(L_1, L_2) + [L_1, L_2]_{\mathfrak{l}}\\
&\langle [L, A_1]', A_2\rangle = \langle \rho(L)A_1, A_2\rangle
\end{aligned}$$

for all $A_1, A_2 \in \mathfrak{a}$ and $L, L_1, L_2 \in \mathfrak{l}$. Obviously, the first two of these conditions and the last one are always satisfied if we choose Ψ as in (25). The third condition is equivalent to the following equations

$$\begin{aligned}
\langle [L_1, L_2]', L_3\rangle &= \langle [\Psi L_1, \Psi L_2]_2, \Psi L_3\rangle &&= \gamma_1(L_1, L_2, L_3)\\
\langle [L_1, L_2]', A\rangle &= \langle [\Psi L_1, \Psi L_2]_2, \Psi A\rangle &&= \langle \alpha_1(L_1, L_2), A\rangle\\
\langle [L_1, L_2]', Z\rangle &= \langle [\Psi L_1, \Psi L_2]_2, \Psi Z\rangle &&= \langle [L_1, L_2]_{\mathfrak{l}}, Z\rangle
\end{aligned}$$

for all $Z \in \mathfrak{l}^*$, $A \in \mathfrak{a}$ and $L_1, L_2, L_3 \in \mathfrak{l}$. The third equation is always satisfied if Ψ is an isometry as in (25). Hence Ψ is a Lie algebra isomorphism if and only if

$$\langle [\Psi L_1, \Psi L_2]_2, \Psi L_3\rangle = \gamma_1(L_1, L_2, L_3) \tag{26}$$

$$\langle [\Psi L_1, \Psi L_2]_2, \Psi A\rangle = \langle \alpha_1(L_1, L_2), A\rangle \tag{27}$$

for all $A \in \mathfrak{a}$ and $L_1, L_2, L_3 \in \mathfrak{l}$. By definition of Ψ Equation (26) is equivalent to

$$\begin{aligned}
&\gamma_1(L_1, L_2, L_3) =\\
&\quad = \Big\langle\, [\bar\sigma(L_1) - \frac{1}{2}\tau^*\tau(L_1) + \tau(L_1) + L_1, \bar\sigma(L_2) - \frac{1}{2}\tau^*\tau(L_2) + \tau(L_2) + L_2]_2,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \bar\sigma(L_3) - \frac{1}{2}\tau^*\tau(L_3) + \tau(L_3) + L_3 \,\Big\rangle\\
&\quad = \Big\langle\, \sigma(L_1, [L_2, \cdot\,]_{\mathfrak{l}}) - \sigma(L_2, [L_1, \cdot\,]_{\mathfrak{l}}) - \frac{1}{2}\tau^*\tau(L_1)([L_2, \cdot\,]_{\mathfrak{l}}) + \frac{1}{2}\tau^*\tau(L_2)([L_1, \cdot\,]_{\mathfrak{l}})\\
&\qquad + \langle \rho(\cdot)\tau(L_1), \tau(L_2)\rangle + \langle \tau(L_1), \alpha_2(L_2, \cdot\,)\rangle - \langle \tau(L_2), \alpha_2(L_1, \cdot\,)\rangle\\
&\qquad + L_1\tau(L_2) - L_2\tau(L_1) + \gamma_2(L_1, L_2, \cdot\,) + \alpha_2(L_1, L_2) + [L_1, L_2]_{\mathfrak{l}},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \bar\sigma(L_3) - \frac{1}{2}\tau^*\tau(L_3) + \tau(L_3) + L_3 \,\Big\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \gamma_2(L_1,L_2,L_3) + \sum_{\text{cycl}} \sigma(L_1,[L_2,L_3]_{\mathfrak{l}}) + \sum_{\text{cycl}} \langle \alpha_2(L_1,L_2), \tau(L_3)\rangle \\
&\quad + \frac{1}{2}\sum_{\text{cycl}} \Big(\langle \tau(L_1), L_2\tau(L_3)\rangle - \langle \tau(L_1), L_3\tau(L_2)\rangle - \langle \tau(L_1), [L_2,L_3]_{\mathfrak{l}}\rangle \Big) \\
&= (\gamma_2 + d\sigma + \langle \alpha_2 \wedge \tau\rangle + \frac{1}{2}\langle \tau \wedge d\tau\rangle)(L_1,L_2,L_3).
\end{aligned}
$$

Moreover, using the previous calculation of $[\Psi(L_1),\Psi(L_2)]_2$ we see that (27) is equivalent to

$$
\begin{aligned}
\langle \alpha_1(L_1,L_2), A\rangle &= \langle [\Psi(L_1),\Psi(L_2)]_2, \Psi(A)\rangle = \langle [\Psi(L_1),\Psi(L_2)]_2, A - \tau^* A\rangle \\
&= \langle L_1\tau(L_2) - L_2\tau(L_1) + \alpha_2(L_1,L_2), A\rangle - \langle [L_1,L_2]_{\mathfrak{l}}, \tau^*(A)\rangle \\
&= \langle \alpha_2(L_1,L_2) + d\tau(L_1,L_2), A\rangle.
\end{aligned}
$$

Hence, Ψ is a Lie algebra isomorphism if and only if $(\alpha_1,\gamma_1) = (\alpha_2,\gamma_2)(\tau,\sigma)$.

Now let us finish the proof of the proposition. If $\mathfrak{d}_{\alpha_1,\gamma_1}(\mathfrak{l},\mathfrak{a},\rho)$ and $\mathfrak{d}_{\alpha_2,\gamma_2}(\mathfrak{l},\mathfrak{a},\rho)$ are equivalent, then we can choose an equivalence map Ψ, which can be written as in (25) and $(\alpha_1,\gamma_1) = (\alpha_2,\gamma_2)(\tau,\sigma)$ holds. Thus $[\alpha_1,\gamma_1] = [\alpha_2,\gamma_2] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$. Conversely, if there exists an element (τ,σ) in $\mathcal{C}^1_Q(\mathfrak{l},\mathfrak{a})$ such that $(\alpha_1,\gamma_1) = (\alpha_2,\gamma_2)(\tau,\sigma)$, then $\Psi(\tau,\sigma) : \mathfrak{d}_{\alpha_1,\gamma_1}(\mathfrak{l},\mathfrak{a},\rho) \to \mathfrak{d}_{\alpha_2,\gamma_2}(\mathfrak{l},\mathfrak{a},\rho)$ (see (25)) is an equivalence. □

Remark 3.2 The map

$$
\begin{array}{rccc}
\overline{\Psi}: & \mathcal{C}^1_Q(\mathfrak{l},\mathfrak{a}) & \longrightarrow & O(\mathfrak{d},\langle\cdot\,,\cdot\rangle) \\
& (\tau,\sigma) & \longmapsto & \Psi(\tau,\sigma)
\end{array}
$$

is an injective group homomorphism. This motivates Definition 2.1. As we have seen above the image of $\overline{\Psi}$ acts on the set of quadratic extensions of $\mathfrak{l}$ by $\mathfrak{a}$ of the form $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$ by equivalences. The orbits of this action are exactly the equivalence classes of such quadratic extensions. Moreover, the map $\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a}) \ni (\alpha,\gamma) \mapsto \mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$ is equivariant with respect to the action of $\mathcal{C}^1_Q(\mathfrak{l},\mathfrak{a})$ on $\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$ and the action of $\mathcal{C}^1_Q(\mathfrak{l},\mathfrak{a})$ on the set $\{\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho) \mid (\alpha,\gamma) \in \mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})\}$ defined by $\overline{\Psi}$. This motivates Formula (6).

Let $(\mathfrak{g},\mathfrak{i},i,p)$ be a quadratic extension of $\mathfrak{l}$ by $\mathfrak{a}$. Then $\mathfrak{g} \cong \mathfrak{i} \oplus \mathfrak{a} \oplus \mathfrak{l} \cong \mathfrak{i}^\perp \oplus \mathfrak{l}$ as vector spaces. Since, furthermore, $\mathfrak{i}$ is isotropic we can choose an isotropic complement $V_{\mathfrak{l}}$ of $\mathfrak{i}^\perp$ in $\mathfrak{g}$ and an isomorphism $s : \mathfrak{l} \to V_{\mathfrak{l}}$ such that $\tilde{p} \circ s = \mathrm{Id}$ holds. Here $\tilde{p} : \mathfrak{g} \to \mathfrak{l}$ is the composition of the natural projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ with p as already defined above.

We define $\alpha \in C^2(\mathfrak{l},\mathfrak{a})$ and $\gamma \in C^3(\mathfrak{l})$ by

$$
\begin{aligned}
i(\alpha(L_1,L_2)) &:= [s(L_1),s(L_2)] - s([L_1,L_2]) + \mathfrak{i} \quad \in \mathfrak{g}/\mathfrak{i} && (28)\\
\gamma(L_1,L_2,L_3) &:= \langle\, [s(L_1),s(L_2)], s(L_3)\rangle\ . && (29)
\end{aligned}
$$

Proposition 3.4 *We have that* $(\alpha,\gamma) \in \mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$. *The quadratic extension* $(\mathfrak{g},\mathfrak{i},i,p)$ *is equivalent to* $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$.

Proof. Here we will denote the inner product and the Lie bracket on $\mathfrak{g}$ by $\langle\cdot\,,\cdot\rangle_{\mathfrak{g}}$ and $[\cdot\,,\cdot]_{\mathfrak{g}}$, respectively. Let $V_{\mathfrak{a}}$ be the orthogonal complement of $\mathfrak{i}\oplus s(\mathfrak{l})$ in $\mathfrak{g}$. Then $\mathfrak{i}^{\perp}=\mathfrak{i}\oplus V_{\mathfrak{a}}$ and we can define a linear map $t:\mathfrak{a}\longrightarrow V_{\mathfrak{a}}$ by

$$i(A)=t(A)+\mathfrak{i}\in\mathfrak{g}/\mathfrak{i}.$$

Since $i:\mathfrak{a}\to\mathfrak{i}^{\perp}/\mathfrak{i}$ is an isometry, also t is an isometry, and because of (11) we have

$$t(\rho(L)A)\equiv[s(L),t(A)]_{\mathfrak{g}}\mod\mathfrak{i}.$$

Recall that $p^*:\mathfrak{l}^*\to\mathfrak{i}$ is an isomorphism satisfying

$$\langle p^*(Z),s(L)\rangle_{\mathfrak{g}}=\langle Z,\tilde{p}\circ s(L)\rangle_{\mathfrak{g}}=Z(L) \tag{30}$$

for all $Z\in\mathfrak{l}^*$ and $L\in\mathfrak{l}$.

Now we consider the triple $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)=(\,\mathfrak{d},[\cdot\,,\cdot],\langle\cdot\,,\cdot\rangle\,)$ for $\alpha\in C^2(\mathfrak{l},\mathfrak{a})$ and $\gamma\in C^3(\mathfrak{l})$ as defined in (28) and (29). We define

$$\Psi=p^*+t+s:\ \mathfrak{d}=\mathfrak{l}^*\oplus\mathfrak{a}\oplus\mathfrak{l}\longrightarrow\mathfrak{g}.$$

By construction $\Psi:(\mathfrak{d},\langle\cdot\,,\cdot\rangle)\to(\mathfrak{g},\langle\cdot\,,\cdot\rangle_{\mathfrak{g}})$ is an isometry. Next we will show that $\Psi[X,Y]=[\Psi(X),\Psi(Y)]_{\mathfrak{g}}$ holds for all $X,Y\in\mathfrak{d}$. To do that we define a bilinear map $[\cdot\,,\cdot]'$ on $\mathfrak{d}\times\mathfrak{d}$ by $[X,Y]'=\Psi^{-1}[\Psi(X),\Psi(Y)]_{\mathfrak{g}}$ and prove that $[\cdot\,,\cdot]'$ satisfies (14)–(17). Clearly, $\langle\cdot\,,\cdot\rangle$ is invariant with respect to $[\cdot\,,\cdot]'$ since $\langle\cdot\,,\cdot\rangle_{\mathfrak{g}}$ is invariant and Ψ is an isometry. By construction of Ψ we have

$$\begin{aligned}
&[\mathfrak{l}^*,\mathfrak{d}]'=\Psi^{-1}[\Psi\mathfrak{l}^*,\Psi\mathfrak{d}]_{\mathfrak{g}}=\Psi^{-1}[\mathfrak{i},\mathfrak{g}]_{\mathfrak{g}}\subset\Psi^{-1}(\mathfrak{i})=\mathfrak{l}^*\\
&[\mathfrak{l}^*,\mathfrak{l}^*]'=\Psi^{-1}[\Psi\mathfrak{l}^*,\Psi\mathfrak{l}^*]_{\mathfrak{g}}=\Psi^{-1}[\mathfrak{i},\mathfrak{i}]_{\mathfrak{g}}=0\\
&[\mathfrak{a},\mathfrak{a}]'=\Psi^{-1}[\Psi\mathfrak{a},\Psi\mathfrak{a}]_{\mathfrak{g}}\subset\Psi^{-1}[\mathfrak{i}^{\perp},\mathfrak{i}^{\perp}]_{\mathfrak{g}}\subset\Psi^{-1}(\mathfrak{i})=\mathfrak{l}^*,
\end{aligned} \tag{31}$$

where we have used the inclusion $[\mathfrak{i}^{\perp},\mathfrak{i}^{\perp}]_{\mathfrak{g}}\subset\mathfrak{i}$, which holds since $\mathfrak{i}^{\perp}/\mathfrak{i}\cong\mathfrak{a}$ and $\mathfrak{a}$ is abelian. Moreover, (28) gives

$$[s(L_1),s(L_2)]_{\mathfrak{g}}-t(\alpha(L_1,L_2))-s([L_1,L_2])\in\mathfrak{i} \tag{32}$$

and by (29) and (30) we have

$$\begin{aligned}
\langle[s(L_1),s(L_2)]_{\mathfrak{g}}-t(\alpha(L_1,L_2))-s([L_1,L_2]_{\mathfrak{l}}),s(L_3)\rangle_{\mathfrak{g}}&=\gamma(L_1,L_2,L_3)\\
&=\langle p^*(\gamma(L_1,L_2,\cdot)),s(L_3)\rangle_{\mathfrak{g}}\,.
\end{aligned} \tag{33}$$

Since $p^*(\gamma(L_1,L_2,\cdot))\in\mathfrak{i}$ and $\langle\cdot\,,\cdot\rangle_{\mathfrak{g}}|_{\mathfrak{i}\times s(\mathfrak{l})}$ is a non-degenerate pairing we obtain by (32) and (33)

$$\begin{aligned}
[s(L_1),s(L_2)]_{\mathfrak{g}}&=p^*(\gamma(L_1,L_2,\cdot))+t(\alpha(L_1,L_2))+s([L_1,L_2]_{\mathfrak{l}})\\
&=\Psi(\gamma(L_1,L_2,\cdot)+\alpha(L_1,L_2)+[L_1,L_2]_{\mathfrak{l}})\,.
\end{aligned}$$

This yields

$$\begin{aligned}
[L_1,L_2]'&=\Psi^{-1}([\Psi(L_1),\Psi(L_2)]_{\mathfrak{g}})=\Psi^{-1}([s(L_1),s(L_2)]_{\mathfrak{g}})\\
&=\gamma(L_1,L_2,\cdot\,)+\alpha(L_1,L_2)+[L_1,L_2]_{\mathfrak{l}}.
\end{aligned} \tag{34}$$

Finally, we have

$$\begin{aligned}\langle [L,A_1]',A_2\rangle &= \langle \Psi([L,A_1]'),\Psi(A_2)\rangle_{\mathfrak{g}} = \langle [s(L),t(A_1)]_{\mathfrak{g}},t(A_2)\rangle_{\mathfrak{g}}\\ &= \langle t(\rho(L)(A_1)),t(A_2)\rangle_{\mathfrak{g}} = \langle \rho(L)(A_1),A_2\rangle.\end{aligned}$$

By Proposition 3.1 this equation together with (31) and (34) implies $[\cdot\,,\cdot]' = [\cdot\,,\cdot]$, thus $\Psi[X,Y] = [\Psi(X),\Psi(Y)]_{\mathfrak{g}}$ for all $X,Y\in\mathfrak{d}$. In particular, $(\mathfrak{d},[\cdot\,,\cdot])$ is a Lie algebra since $(\mathfrak{g},[\cdot\,,\cdot]_{\mathfrak{g}})$ is a Lie algebra. Proposition 3.1 now implies $(\alpha,\gamma)\in\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$. Moreover, we conclude that $\Psi:\mathfrak{d}\to\mathfrak{g}$ is an equivalence of the quadratic extensions $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$ and $(\mathfrak{g},\mathfrak{i},i,p)$. $\square$

Corollary 3.1 *The cohomology class $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ does not depend on the choice of s.*

Proof. Let $s_i:\mathfrak{l}\to\mathfrak{g}$, $i=1,2$, be two linear maps with isotropic image and $\tilde p\circ s_i=\mathrm{Id}$. Consider $(\alpha_i,\gamma_i)\in\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$, $i=1,2$, as above. By Proposition 3.4 the quadratic extensions $\mathfrak{d}_{\alpha_i,\gamma_i}(\mathfrak{l},\mathfrak{a},\rho)$, $i=1,2$, are equivalent since both are equivalent to $(\mathfrak{g},\mathfrak{i},i,p)$. Proposition 3.3 now implies $[\alpha_1,\gamma_1]=[\alpha_2,\gamma_2]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$. $\square$

We can summarize the results of Section 3 as follows.

Theorem 3.1 *The equivalence classes of quadratic extensions of a Lie algebra $\mathfrak{l}$ by an orthogonal module $\mathfrak{a}$ are in one-to-one correspondence with elements of $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$.*

4 Balanced extensions

In this section we equip any metric Lie algebra $\mathfrak{g}$ without simple ideals with the structure of a quadratic extension in a canonical way, i.e., we construct a canonical isotropic ideal $\mathfrak{i}(\mathfrak{g})\subset\mathfrak{g}$ such that $\mathfrak{i}(\mathfrak{g})^\perp/\mathfrak{i}(\mathfrak{g})$ is abelian. A quadratic extension $(\mathfrak{g},\mathfrak{i},i,p)$ will be called balanced, if $\mathfrak{i}=\mathfrak{i}(\mathfrak{g})$. The main result of this section is Theorem 4.1 which describes the subset of $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ corresponding to balanced extensions.

In order to construct the desired canonical ideal we need a little preparation. Let $\mathfrak{g}$ be a (real, finite-dimensional) Lie algebra, and let V be a finite-dimensional $\mathfrak{g}$-module. The socle $S(V)\subset V$ is by definition the maximal submodule of V on which $\mathfrak{g}$ acts semi-simply. It is equal to the sum of all irreducible submodules of V. There is the dual notion of the radical $R(V)\subset V$ which is the minimal submodule such that $\mathfrak{g}$ acts semi-simply on $V/R(V)$. For later use we collect the basic computation rules for the functors S and R. Let $U,V\subset W$ be $\mathfrak{g}$-submodules. Then

$$\begin{array}{ll} S(W)=0\Rightarrow W=0 & R(W)=W\Rightarrow W=0\\ S(S(W))=S(W) & R(S(W))=0\\ S(U\cap V)=S(U)\cap V & R(U\cap V)\subset R(U)\cap R(V)\\ S(U+V)\supset S(U)+S(V) & R(U+V)=R(U)+R(V)\\ S(W/U)\subset (S(W)+U)/U & R(W/U)=(R(W)+U)/U\ . \end{array} \tag{35}$$

Definition 4.1 *We define the higher socles $S_k(V) \subset V$ and radicals $R_k(V) \subset V$, $k \in \mathbb{N}$, inductively by*

$$\begin{aligned} S_0(V) &:= \{0\} \ , \quad S_k(V) := (p_{k-1})^{-1}(S(V/S_{k-1}(V))) \ , \\ R_0(V) &:= V \ , \quad R_k(V) := R(R_{k-1}(V)) \ , \end{aligned}$$

where $p_{k-1} : V \to V/S_{k-1}(V)$ is the natural projection.

Clearly, $S_1(V) = S(V)$ and $R_1(V) = R(V)$.

If V^* is the dual $\mathfrak{g}$-module, then for $k \in \mathbb{N}$

$$S_k(V^*) = R_k(V)^\perp \quad \text{and} \quad R_k(V^*) = S_k(V)^\perp \ . \tag{36}$$

We are particularly interested in the case of the adjoint representation $V = \mathfrak{g}$ of $\mathfrak{g}$. In this case Definition 4.1 provides an increasing and a decreasing chain of ideals of $\mathfrak{g}$

$$\begin{aligned} \{0\} = S_0(\mathfrak{g}) \subset S_1(\mathfrak{g}) \subset S_2(\mathfrak{g}) \subset \quad \ldots \quad \subset S_{l_+}(\mathfrak{g}) = \mathfrak{g} \\ \mathfrak{g} = R_0(\mathfrak{g}) \supset R_1(\mathfrak{g}) \supset R_2(\mathfrak{g}) \supset \quad \ldots \quad \supset R_{l_-}(\mathfrak{g}) = \{0\} \ . \end{aligned}$$

We call $R(\mathfrak{g})$ the radical of nilpotency of $\mathfrak{g}$ in order to distinguish it from the (solvable) radical $\mathfrak{r}$ and the nilpotent radical (= maximal nilpotent ideal) $\mathfrak{n}$. Note that for $k > 1$ the radical of nilpotency of the Lie algebra $R_{k-1}(\mathfrak{g})$ may be larger than $R_k(\mathfrak{g})$ since $R_k(\mathfrak{g})$ is defined in terms of the $\mathfrak{g}$-module structure of $R_{k-1}(\mathfrak{g})$. It is a consequence of Lie's Theorem that (see [Bou 71])

$$R(\mathfrak{g}) = \mathfrak{r} \cap \mathfrak{g}' = [\mathfrak{r}, \mathfrak{g}'] \subset \mathfrak{n} \tag{37}$$

and that $R(\mathfrak{g})$ acts trivially on any semi-simple $\mathfrak{g}$-module V. The last property implies (consider $V = S_k(\mathfrak{g})/S_{k-1}(\mathfrak{g})$)

$$[R(\mathfrak{g}), S_k(\mathfrak{g})] \subset S_{k-1}(\mathfrak{g}) \ . \tag{38}$$

We will also need the relation of $\mathfrak{z}(\mathfrak{g})$ with $S(\mathfrak{g})$. Of course, $\mathfrak{z}(\mathfrak{g}) \subset S(\mathfrak{g})$. We will formulate a more precise result for the case that $\mathfrak{g}$ does not contain simple ideals which is relevant for quadratic extensions (see Lemma 3.1). Of course, the general case does not present essential difficulties since any Lie algebra splits into a direct sum of a semi-simple ideal and an ideal which does not contain simple ideals.

Lemma 4.1 *If $\mathfrak{g}$ does not contain a simple ideal, then*

(a) $S(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) + S(\mathfrak{g}) \cap R(\mathfrak{g})$.

(b) $S(\mathfrak{g}) \subset R(\mathfrak{g})$ *if and only if* $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{g}'$.

Proof. Let $\mathfrak{s}$ be a $\mathfrak{g}$-invariant complement of $S(\mathfrak{g}) \cap R(\mathfrak{g})$ in $S(\mathfrak{g})$. Note that $\mathfrak{s}$ is a reductive Lie algebra. Let $p : \mathfrak{g} \to \mathfrak{g}/R(\mathfrak{g})$ be the projection, and let $\mathfrak{t}$ be a $\mathfrak{g}$-invariant complement of $p(S(\mathfrak{g})) = p(\mathfrak{s})$ in $\mathfrak{g}/R(\mathfrak{g})$. We obtain a decomposition of $\mathfrak{g}$ into a direct sum of ideals

$$\mathfrak{g} = \mathfrak{s} \oplus p^{-1}(\mathfrak{t}) = \mathfrak{z}(\mathfrak{s}) \oplus \mathfrak{s}' \oplus p^{-1}(\mathfrak{t}) \ .$$

The decomposition shows that $\mathfrak{z}(\mathfrak{s}) \subset \mathfrak{z}(\mathfrak{g})$. Now, if $\mathfrak{g}$ does not contain simple ideals, then $\mathfrak{s}' = 0$. We obtain $S(\mathfrak{g}) = \mathfrak{s} \oplus S(\mathfrak{g}) \cap R(\mathfrak{g}) \subset \mathfrak{z}(\mathfrak{g}) + S(\mathfrak{g}) \cap R(\mathfrak{g})$. Since the opposite inclusion is obvious this proves (a).

Assume that $S(\mathfrak{g}) \subset R(\mathfrak{g})$. Using (37) we find $\mathfrak{z}(\mathfrak{g}) \subset S(\mathfrak{g}) \subset R(\mathfrak{g}) \subset \mathfrak{g}'$. If $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{g}'$, then $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{r} \cap \mathfrak{g}' = R(\mathfrak{g})$. Now (a) implies that $S(\mathfrak{g}) \subset R(\mathfrak{g})$. $\square$

Definition 4.2 *Let $\mathfrak{g}$ be a Lie algebra. We define characteristic ideals $\mathfrak{i}(\mathfrak{g}) \subset \mathfrak{j}(\mathfrak{g}) \subset \mathfrak{g}$ by*

$$\mathfrak{i}(\mathfrak{g}) := \sum_{k=1}^{\infty} R_k(\mathfrak{g}) \cap S_k(\mathfrak{g}) \quad \text{and} \quad \mathfrak{j}(\mathfrak{g}) := \bigcap_{k=1}^{\infty} (R_k(\mathfrak{g}) + S_k(\mathfrak{g})) \ .$$

Of course, the sum and the intersection are only formally infinite. For all $j \le k \le l$ we have $R_k(\mathfrak{g}) \cap S_k(\mathfrak{g}) \subset R_j(\mathfrak{g})$ and $R_k(\mathfrak{g}) \cap S_k(\mathfrak{g}) \subset S_l(\mathfrak{g})$. Thus $R_k(\mathfrak{g}) \cap S_k(\mathfrak{g}) \subset R_l(\mathfrak{g}) + S_l(\mathfrak{g})$ for all l. This shows that indeed $\mathfrak{i}(\mathfrak{g}) \subset \mathfrak{j}(\mathfrak{g})$.

Lemma 4.2 *The ideals $\mathfrak{i}(\mathfrak{g})$ and $\mathfrak{j}(\mathfrak{g})$ satisfy*

(a) $\mathfrak{j}(\mathfrak{g}) = S(\mathfrak{g}) + \sum_{k=1}^{\infty} R_k(\mathfrak{g}) \cap S_{k+1}(\mathfrak{g})$.

(b) *The natural representation of $\mathfrak{g}$ on the quotient $\mathfrak{j}(\mathfrak{g})/\mathfrak{i}(\mathfrak{g})$ is semi-simple.*

(c) $[R(\mathfrak{g}), \mathfrak{j}(\mathfrak{g})] \subset \mathfrak{i}(\mathfrak{g})$.

(d) *If $\mathfrak{g}$ does not contain a simple ideal, then the Lie algebra $\mathfrak{j}(\mathfrak{g})/\mathfrak{i}(\mathfrak{g})$ is abelian.*

Proof. We compute

$$\begin{aligned}
\mathfrak{j}(\mathfrak{g}) &= \bigcap_{k=1}^{\infty} (S_k(\mathfrak{g}) + R_k(\mathfrak{g})) = S(\mathfrak{g}) + R(\mathfrak{g}) \cap \bigcap_{k=2}^{\infty} (S_k(\mathfrak{g}) + R_k(\mathfrak{g})) \\
&= S(\mathfrak{g}) + R(\mathfrak{g}) \cap \left(S_2(\mathfrak{g}) + R_2(\mathfrak{g}) \cap \bigcap_{k=3}^{\infty} (S_k(\mathfrak{g}) + R_k(\mathfrak{g})) \right) \\
&= S(\mathfrak{g}) + R(\mathfrak{g}) \cap S_2(\mathfrak{g}) + R_2(\mathfrak{g}) \cap \bigcap_{k=3}^{\infty} (S_k(\mathfrak{g}) + R_k(\mathfrak{g})) \\
&= \ldots \\
&= S(\mathfrak{g}) + \sum_{k=1}^{\infty} R_k(\mathfrak{g}) \cap S_{k+1}(\mathfrak{g}) \ .
\end{aligned}$$

This shows (a).

In order to prove (b) we show that $R(\mathfrak{j}(\mathfrak{g})) \subset \mathfrak{i}(\mathfrak{g})$, where the radical is taken with respect to the $\mathfrak{g}$-module structure. Using (a) and the rules (35) we find

$$\begin{aligned}
R(\mathfrak{j}(\mathfrak{g})) &= R(S(\mathfrak{g})) + \sum_{k=1}^{\infty} R\left(R_k(\mathfrak{g}) \cap S_{k+1}(\mathfrak{g})\right) \subset \sum_{k=1}^{\infty} R(R_k(\mathfrak{g})) \cap S_{k+1}(\mathfrak{g}) \\
&= \sum_{k=1}^{\infty} R_{k+1}(\mathfrak{g}) \cap S_{k+1}(\mathfrak{g}) \subset \mathfrak{i}(\mathfrak{g}) \ .
\end{aligned}$$

This proves (b) and implies that $R(\mathfrak{g})$ acts trivially on $\mathfrak{j}(\mathfrak{g})/\mathfrak{i}(\mathfrak{g})$. Alternatively, (c) could be shown directly using (a) and (38).

By (a) we have $\mathfrak{j}(\mathfrak{g}) \subset S(\mathfrak{g}) + R(\mathfrak{g})$. If $\mathfrak{g}$ does not contain simple ideals, then Lemma 4.1 implies that $\mathfrak{j}(\mathfrak{g}) \subset \mathfrak{z}(\mathfrak{g}) + R(\mathfrak{g})$. Now (d) follows from (c). $\square$

From now on let $\mathfrak{g}$ be a metric Lie algebra. Then by (36) we have

$$R_k(\mathfrak{g})^\perp = S_k(\mathfrak{g}) \ .$$

This implies that $\mathfrak{i}(\mathfrak{g})^\perp = \mathfrak{j}(\mathfrak{g})$. In particular, $\mathfrak{i}(\mathfrak{g})$ is isotropic.

Definition 4.3 *If $\mathfrak{g}$ is a metric Lie algebra, then we call $\mathfrak{i}(\mathfrak{g})$ its canonical isotropic ideal.*

Observe that both $\mathfrak{i}(\mathfrak{g})$ and $\mathfrak{i}(\mathfrak{g})^\perp = \mathfrak{j}(\mathfrak{g})$ are completely determined by the Lie algebra structure of $\mathfrak{g}$ and do not depend on the particular form of the inner product on $\mathfrak{g}$.

Let $\mathfrak{l}$ be a Lie algebra, and let $\mathfrak{a}$ be an orthogonal $\mathfrak{l}$-module.

Definition 4.4 *A quadratic extension $(\mathfrak{g}, \mathfrak{i}, i, p)$ of $\mathfrak{l}$ by $\mathfrak{a}$ is called balanced if $\mathfrak{i} = \mathfrak{i}(\mathfrak{g})$. We call the extension regularly balanced, if in addition $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{g}'$.*

The study of metric Lie algebras is easily reduced to the study of metric Lie algebras without simple ideals. Indeed, each simple ideal of a metric Lie algebra is non-degenerate (see e.g. [KO 02], Lemma 2.2). It follows that any metric Lie algebra is the direct sum of a semi-simple metric Lie algebra and a metric Lie algebra without simple ideals.

Proposition 4.1 *Any metric Lie algebra $\mathfrak{g}$ without simple ideals has the structure of a balanced quadratic extension in a canonical way. It is regularly balanced if and only if $\mathfrak{g}$ does not contain a non-degenerate abelian ideal.*

Proof. By Part (d) of Lemma 4.2 the quotient $\mathfrak{i}(\mathfrak{g})^\perp/\mathfrak{i}(\mathfrak{g})$ is abelian. Thus $\mathfrak{i}(\mathfrak{g}) \subset \mathfrak{g}$ defines a canonical quadratic extension (see (12)) which is balanced by the very definition.

Any non-degenerate abelian ideal of $\mathfrak{g}$ is central. Since $\mathfrak{z}(\mathfrak{g})^\perp = \mathfrak{g}'$ such an ideal does not intersect $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}'$. Vice versa, any complement of $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}'$ in $\mathfrak{z}(\mathfrak{g})$ is a non-degenerate abelian ideal. This proves the second assertion. $\square$

Remark 4.1 Slightly different canonical isotropic ideals (and corresponding notions of balanced extensions) can be obtained from other canonical decreasing chains of ideals of $\mathfrak{g}$ (like the derived series of $\mathfrak{r}$ or the lower central series of $\mathfrak{n}$) and the corresponding increasing chains of their orthogonal complements. In fact, in an earlier stage of our work we investigated metric Lie algebras based on the series

$$\mathfrak{g} \supset \mathfrak{r} \supset \mathfrak{r}' \supset (\mathfrak{r}')^2 \ldots \supset (\mathfrak{r}')^k \supset \ldots \ .$$

For related constructions compare also [Bor 97] and [N 03]. We are indebted to L. Berard Bergery who drawed our attention to the series used here, which is also the basis of his investigations of pseudo-Riemanniann holonomy representations and symmetric spaces (compare [BB1] and [BB2]) and which seems to be most appropriate for the study of metric Lie algebras.

Now we are going to derive necessary and sufficient conditions for a quadratic extension to be (regularly) balanced in terms of the characterizing data $\mathfrak{l}$, ρ, and $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$. We will work with the standard model $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$, $(\alpha,\gamma)\in\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$, of such an extension.

A first necessary condition is given by Part (b) of Lemma 4.2.

Corollary 4.1 *If $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$ is balanced, then the representation ρ of $\mathfrak{l}$ on $\mathfrak{a}$ is semi-simple. In particular, $\rho|_{R(\mathfrak{l})}=0$.*

We look at the decreasing chain of ideals

$$\mathfrak{l}=R_0(\mathfrak{l})\supset R_1(\mathfrak{l})\supset R_2(\mathfrak{l})\supset\ldots\supset R_k(\mathfrak{l})\supset\ldots\ .$$

Then we can form corresponding quadratic extensions

$$\mathfrak{d}_k:=\mathfrak{d}_{\alpha_k,\gamma_k}(\mathfrak{a},R_k(\mathfrak{l}),\rho_k)\ ,\quad k\ge 0\ ,$$

where ρ_k, α_k and γ_k are obtained from ρ, α, γ by restriction to $R_k(\mathfrak{l})$. As a vector space we have

$$\mathfrak{d}_k=R_k(\mathfrak{l})^*\oplus\mathfrak{a}\oplus R_k(\mathfrak{l})\ .$$

The Lie algebra $\mathfrak{d}_k$ is equipped with a natural action of $\mathfrak{d}=\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$ by antisymmetric derivations. Indeed, the subspace

$$\mathfrak{h}_k:=\mathfrak{l}^*\oplus\mathfrak{a}\oplus R_k(\mathfrak{l})\subset\mathfrak{d}$$

is a coisotropic ideal of $\mathfrak{d}$. Observe that $\mathfrak{h}_k^\perp$ is equal to the annihilator $R_k(\mathfrak{l})^\perp=S_k(\mathfrak{l}^*)$ of $R_k(\mathfrak{l})$ in $\mathfrak{l}^*$. It follows $\mathfrak{d}_k\cong\mathfrak{h}_k/\mathfrak{h}_k^\perp$ as $\mathfrak{d}$-module.

The subspace

$$M_k:=R_k(\mathfrak{l})^*\oplus\mathfrak{a}\subset\mathfrak{d}_k$$

is a $\mathfrak{d}$-submodule and the projection onto the second summand

$$pr_{\mathfrak{a}}:M_k\to\mathfrak{a}$$

is $\mathfrak{d}$-equivariant.

Now we consider the following conditions (all socles are taken with respect to the $\mathfrak{d}$-module structure):

(a_k) $\quad S(\mathfrak{d}_k)\subset M_k$

(b_k) $\quad pr_{\mathfrak{a}}(S(M_k))\subset\mathfrak{a}$ is non-degenerate w.r.t. $\langle\cdot\,,\cdot\rangle_{\mathfrak{a}}$

(b_0') $\quad pr_{\mathfrak{a}}(S(M_0))=0$.

Then we have the following

Lemma 4.3 *The quadratic extension $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$ is balanced if and only if the conditions (a_k) and (b_k) hold for all $k \ge 0$. The assertion remains true if we replace "balanced" by "regularly balanced" and (b_0) by (b'_0).*

Proof. For $k \ge 0$ we introduce the following ideals of $\mathfrak{d}$:

$$\begin{aligned}
\mathfrak{i}_k &:= \sum_{l=1}^{k} R_l(\mathfrak{d}) \cap S_l(\mathfrak{d}) = S_k(\mathfrak{d}) \cap \mathfrak{i}(\mathfrak{d}) \ , \\
\mathfrak{j}_k &:= \mathfrak{i}_k^{\perp} = R_k(\mathfrak{d}) + \mathfrak{j}(\mathfrak{d}) = R_k(\mathfrak{d}) + \mathfrak{m}_k \ , \quad \text{where} \\
\mathfrak{m}_k &:= \sum_{l=1}^{k} R_{l-1}(\mathfrak{d}) \cap S_l(\mathfrak{d}) \supset \mathfrak{i}_k \ .
\end{aligned}$$

We are interested in the socle and the radical of the $\mathfrak{d}$-module $\mathfrak{j}_k/\mathfrak{i}_k$. We will frequently use the rules (35). First we have

$$R(\mathfrak{m}_k) = \sum_{l=1}^{k} R(R_{l-1}(\mathfrak{d}) \cap S_l(\mathfrak{d})) \subset \sum_{l=1}^{k} R_l(\mathfrak{d}) \cap R(S_l(\mathfrak{d})) \subset \mathfrak{i}_k \ .$$

This implies

$$R(\mathfrak{j}_k/\mathfrak{i}_k) = (R(\mathfrak{j}_k) + \mathfrak{i}_k)/\mathfrak{i}_k = (R_{k+1}(\mathfrak{d}) + R(\mathfrak{m}_k) + \mathfrak{i}_k)/\mathfrak{i}_k = (R_{k+1}(\mathfrak{d}) + \mathfrak{i}_k)/\mathfrak{i}_k$$

and

$$S(\mathfrak{j}_k/\mathfrak{i}_k) = R(\mathfrak{j}_k/\mathfrak{i}_k)^{\perp} = (S_{k+1}(\mathfrak{d}) \cap \mathfrak{j}_k)/\mathfrak{i}_k = \mathfrak{m}_{k+1}/\mathfrak{i}_k \ . \tag{39}$$

It follows that

$$S(\mathfrak{j}_k/\mathfrak{i}_k) + R(\mathfrak{j}_k/\mathfrak{i}_k) = (R_{k+1}(\mathfrak{d}) + \mathfrak{m}_{k+1})/\mathfrak{i}_k = \mathfrak{j}_{k+1}/\mathfrak{i}_k$$

and

$$S(\mathfrak{j}_k/\mathfrak{i}_k) \cap R(\mathfrak{j}_k/\mathfrak{i}_k) = (S(\mathfrak{j}_k/\mathfrak{i}_k) + R(\mathfrak{j}_k/\mathfrak{i}_k))^{\perp} = \mathfrak{i}_{k+1}/\mathfrak{i}_k \ . \tag{40}$$

For $k \ge 0$ we consider the condition

$$(c_k) \qquad \mathfrak{h}_k = \mathfrak{j}_k \ .$$

Note that (c_0) is trivially satisfied. We now claim

$$(a_k), (b_k), (c_k) \Rightarrow (c_{k+1}) \tag{41}$$

Let us prove (41). We fix $k \ge 0$ and assume (a_k), (b_k), and (c_k). Condition (c_k) implies that $\mathfrak{h}_k^{\perp} = \mathfrak{i}_k$ and $\mathfrak{d}_k = \mathfrak{h}_k/\mathfrak{h}_k^{\perp} = \mathfrak{j}_k/\mathfrak{i}_k$. This together with (a_k) yields

$$S(\mathfrak{j}_k/\mathfrak{i}_k) = S(\mathfrak{d}_k) \subset M_k = (\mathfrak{l}^* \oplus \mathfrak{a})/\mathfrak{i}_k \ .$$

By (39) we obtain

$$\mathfrak{m}_{k+1} \subset \mathfrak{l}^* \oplus \mathfrak{a} \ . \tag{42}$$

By (a_k) we have $S(M_k) = S(\mathfrak{d}_k)$. Now (b_k) tells us that

$$\mathfrak{l}^*/\mathfrak{i}_k \supset S(\mathfrak{d}_k) \cap S(\mathfrak{d}_k)^\perp = S(\mathfrak{d}_k) \cap R(\mathfrak{d}_k) = S(\mathfrak{j}_k/\mathfrak{i}_k) \cap R(\mathfrak{j}_k/\mathfrak{i}_k)\ .$$

By (40) we obtain

$$\mathfrak{i}_{k+1} \subset \mathfrak{l}^*\ . \tag{43}$$

Taking orthogonal complements this gives

$$\mathfrak{j}_{k+1} \supset \mathfrak{l}^* \oplus \mathfrak{a}\ . \tag{44}$$

Let $pr_\mathfrak{l} : \mathfrak{d} \to \mathfrak{l}$ be the natural projection. Using (42) we obtain

$$pr_\mathfrak{l}(\mathfrak{j}_{k+1}) = pr_\mathfrak{l}(R_{k+1}(\mathfrak{d}) + \mathfrak{m}_{k+1}) = pr_\mathfrak{l}(R_{k+1}(\mathfrak{d}))\ .$$

Now $pr_\mathfrak{l}(R_{k+1}(\mathfrak{d})) = R_{k+1}(\mathfrak{l})$. This together with (44) shows that

$$\mathfrak{j}_{k+1} = \mathfrak{l}^* \oplus \mathfrak{a} \oplus R_{k+1}(\mathfrak{l}) = \mathfrak{h}_{k+1}\ .$$

This is (c_{k+1}), thus we have proved the claim (41).

Now assume that $\mathfrak{d}$ satisfies (a_k) and (b_k) for all $k \in \mathbb{N}_0$. Then by (41) and the triviality of (c_0) Condition (c_k) holds for any k. If k is sufficiently large, then $\mathfrak{j}_k = \mathfrak{j}(\mathfrak{d})$ and $\mathfrak{h}_k = \mathfrak{l}^* \oplus \mathfrak{a}$. We obtain $\mathfrak{l}^* \oplus \mathfrak{a} = \mathfrak{j}(\mathfrak{d})$ or, equivalently, $\mathfrak{l}^* = \mathfrak{i}(\mathfrak{d})$. Thus $\mathfrak{d}$ is balanced.

For the opposite direction we first recall that $\mathfrak{m}_{k+1} \subset \mathfrak{j}(\mathfrak{d})$ and $\mathfrak{i}_{k+1} \subset \mathfrak{i}(\mathfrak{d})$. Therefore, if $\mathfrak{d}$ is balanced, then (42) and (43) hold. We assume in addition that (c_k) holds. Then

$$S(\mathfrak{d}_k) = S(\mathfrak{j}_k/\mathfrak{i}_k) = \mathfrak{m}_{k+1}/\mathfrak{i}_k$$

and

$$S(\mathfrak{d}_k) \cap R(\mathfrak{d}_k) = S(\mathfrak{j}_k/\mathfrak{i}_k) \cap S(\mathfrak{j}_k/\mathfrak{i}_k)^\perp = \mathfrak{i}_{k+1}/\mathfrak{i}_k\ .$$

Now (42) implies (a_k), in particular $S(M_k) = S(\mathfrak{j}_k/\mathfrak{i}_k)$. This together with (43) yields $S(M_k) \cap S(M_k)^\perp \subset \mathfrak{l}^*/\mathfrak{i}_k$, hence (b_k). Thus for any $k \geq 0$ the following implication is true

$$\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}, \rho) \text{ balanced and } (c_k) \Rightarrow (a_k), (b_k)\ .$$

Since (c_0) is the empty condition we obtain using (41) that for a balanced quadratic extension (a_k), (b_k) hold for all k.

Assume now that the conditions (a_k), (b_k), and the strengthend version (b_0') of (b_0) hold. Then $\mathfrak{d}$ is balanced and $S(\mathfrak{d}) \subset \mathfrak{l}^*$ which implies that $S(\mathfrak{d})$ is isotropic, hence $S(\mathfrak{d}) \subset R(\mathfrak{d})$. By Assertion (b) of Lemma 4.1 the extension $\mathfrak{d}$ is regularly balanced. Vice versa, if $\mathfrak{d}$ is regularly balanced, then $S(\mathfrak{d}) \subset R(\mathfrak{d})$. Hence $S(\mathfrak{d}) = S(\mathfrak{d}) \cap R(\mathfrak{d}) \subset \mathfrak{i}(\mathfrak{g}) = \mathfrak{l}^*$, i.e., Condition (b_0') holds. This finishes the proof of the lemma. □

Remark 4.2 For sufficiently large k condition (b_k) simply says that $S(\mathfrak{a})$ is non-degenerate. Thus $\mathfrak{a} = S(\mathfrak{a}) \oplus S(\mathfrak{a})^\perp$. Taking socles we obtain $S\left(S(\mathfrak{a})^\perp\right) = 0$, hence $S(\mathfrak{a})^\perp = 0$. It follows that $\mathfrak{a}$ is a semi-simple $\mathfrak{l}$-module. We just recover Corollary 4.1.

Theorem 4.1 *Let $\mathfrak{l}$ be a Lie algebra, let $(\rho, \mathfrak{a}, \langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ be an orthogonal $\mathfrak{l}$-module, and let $(\alpha,\gamma) \in \mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$. If ρ is semi-simple, then $\mathfrak{a} = \mathfrak{a}^{\mathfrak{l}} \oplus \rho(\mathfrak{l})\mathfrak{a}$, and we have a corresponding decomposition $\alpha = \alpha_0 + \alpha_1$. In this case we consider the following conditions*

(A_0) *Let $L_0 \in \mathfrak{z}(\mathfrak{l}) \cap \ker\rho$ be such that there exist elements $A_0 \in \mathfrak{a}$ and $Z_0 \in \mathfrak{l}^*$ satisfying for all $L \in \mathfrak{l}$*

(i) $\alpha(L, L_0) = \rho(L)A_0$,

(ii) $\gamma(L, L_0, \cdot) = -\langle A_0, \alpha(L,\cdot)\rangle_{\mathfrak{a}} + \langle Z_0, [L,\cdot]_{\mathfrak{l}}\rangle$ *as an element of* $\mathfrak{l}^*$,

then $L_0 = 0$.

(B_0) *The subspace $\alpha_0(\ker\,[\cdot\,,\cdot]_{\mathfrak{l}}) \subset \mathfrak{a}^{\mathfrak{l}}$ is non-degenerate.*

(B_0') $\alpha_0(\ker\,[\cdot\,,\cdot]_{\mathfrak{l}}) = \mathfrak{a}^{\mathfrak{l}}$.

(A_k) $(k \geq 1)$
Let $\mathfrak{k} \subset S(\mathfrak{l}) \cap R_k(\mathfrak{l})$ be an $\mathfrak{l}$-ideal such that there exist elements $\Phi_1 \in \mathrm{Hom}(\mathfrak{k},\mathfrak{a})$ and $\Phi_2 \in \mathrm{Hom}(\mathfrak{k}, R_k(\mathfrak{l})^*)$ satisfying for all $L \in \mathfrak{l}$ and $K \in \mathfrak{k}$

(i) $\alpha(L,K) = \rho(L)\Phi_1(K) - \Phi_1([L,K]_{\mathfrak{l}})$,

(ii) $\gamma(L,K,\cdot) = -\langle \Phi_1(K), \alpha(L,\cdot)\rangle_{\mathfrak{a}} + \langle \Phi_2(K), [L,\cdot]_{\mathfrak{l}}\rangle + \langle \Phi_2([L,K]_{\mathfrak{l}}),\cdot\rangle$ *as an element of* $R_k(\mathfrak{l})^*$,

then $\mathfrak{k} = 0$.

(B_k) $(k \geq 1)$
Let $\mathfrak{b}_k \subset \mathfrak{a}$ be the maximal submodule such that the system of equations

$$\langle \alpha(L,K), B\rangle_{\mathfrak{a}} = \langle \rho(L)\Phi(K) - \Phi([L,K]_{\mathfrak{l}}), B\rangle_{\mathfrak{a}}\ , \quad L \in \mathfrak{l}, K \in R_k(\mathfrak{l}), B \in \mathfrak{b}_k,$$

has a solution $\Phi \in \mathrm{Hom}(R_k(\mathfrak{l}),\mathfrak{a})$. Then $\mathfrak{b}_k$ is non-degenerate.

Let m be such that $R_{m+1}(\mathfrak{l}) = 0$. Then the quadratic extension $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$ is balanced if and only if ρ is semi-simple and the conditions (A_k) and (B_k) hold for all $0 \leq k \leq m$. The assertion remains true if we replace "balanced" by "regularly balanced" and (B_0) by (B_0').

Proof. For $k \geq m+1$ condition (a_k) is trivially satisfied and (b_k) is equivalent to the semi-simplicity of ρ (see Remark 4.2). For $0 \leq k \leq m$ we will show assuming ρ to be semi-simple that (A_k), (B_0'), and (B_k) are equivalent to (a_k), (b_0'), and (b_k), respectively. The theorem then follows from Lemma 4.3.

First we consider the case $k = 0$. Note that for any Lie algebra $\mathfrak{g}$

$$S(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, S(\mathfrak{g})]\ .$$

Lemma 4.4 *Let $\mathfrak{g}$ be a metric Lie algebra and $\mathfrak{j} \subset \mathfrak{g}$ be a nilpotent ideal. Then $[\mathfrak{g}, S(\mathfrak{g})] \subset \mathfrak{j}^{\perp}$.*

Proof. The Lie algebra $\mathfrak{j}$ acts nilpotently and semi-simply, hence trivially, on $S(\mathfrak{g})$. We obtain

$$\langle [\mathfrak{g}, S(\mathfrak{g})], \mathfrak{j}\rangle = \langle \mathfrak{g}, [\mathfrak{j}, S(\mathfrak{g})]\rangle = 0 \ .$$

$\square$

Applying the lemma to the nilpotent ideal $M_0 \subset \mathfrak{d} = \mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}, \rho)$ we obtain $\mathfrak{z}(\mathfrak{d}) \subset S(\mathfrak{d}) \subset \mathfrak{z}(\mathfrak{d}) + \mathfrak{l}^*$. Note that $S(M_0) = S(\mathfrak{d}) \cap M_0$. Therefore we can reformulate the conditions for $k = 0$ in terms of $\mathfrak{z}(\mathfrak{d})$:

(a_0)	$\mathfrak{z}(\mathfrak{d}) \subset M_0$
(b_0)	$pr_\mathfrak{a}(\mathfrak{z}(\mathfrak{d}) \cap M_0) \subset \mathfrak{a}$ is non-degenerate w.r.t. $\langle \cdot\,, \cdot\rangle_\mathfrak{a}$
(b_0')	$pr_\mathfrak{a}(\mathfrak{z}(\mathfrak{d}) \cap M_0) = 0$.

Using the commutator formulas (16) to (21) we find by straightforward computation that an element $Z_0 - A_0 + L_0 \in \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l} = \mathfrak{d}$ is central if and only if the equations (i), (ii) of condition (A_0) are satisfied and $L_0 \in \mathfrak{z}(\mathfrak{l}) \cap \ker(\rho)$. This shows the equivalence of (a_0) and (A_0). Moreover, it implies that $pr_\mathfrak{a}(\mathfrak{z}(\mathfrak{d}) \cap M_0)$ consists of all elements $A_0 \in \mathfrak{a}^\mathfrak{l}$ such that the linear functional on $\Lambda^2(\mathfrak{l})$ given by $\langle A_0, \alpha(\cdot, \cdot)\rangle = \langle A_0, \alpha_0(\cdot, \cdot)\rangle$ factors over the commutator map $[\cdot\,, \cdot]_\mathfrak{l} : \Lambda^2(\mathfrak{l}) \to \mathfrak{l}$, i.e., it is given by the composition of a linear functional Z_0 on $\mathfrak{l}$ with $[\cdot\,, \cdot]_\mathfrak{l}$. It follows that $pr_\mathfrak{a}(\mathfrak{z}(\mathfrak{d}) \cap M_0) \perp \alpha_0(\ker [\cdot\,, \cdot]_\mathfrak{l})$. Vice versa, if $A_0 \in \alpha_0(\ker [\cdot\,, \cdot]_\mathfrak{l})^{\perp_{\mathfrak{a}^\mathfrak{l}}}$, then we can unambigously define an element $Z_1 \in (\mathfrak{l}')^*$ by

$$\langle Z_1, [L_1, L_2]\rangle = \langle A_0, \alpha_0(L_1, L_2)\rangle_\mathfrak{a} \ , \quad L_1, L_2 \in \mathfrak{l} \ ,$$

which can be extended to $\mathfrak{l}$ in an arbitrary manner. We conclude that

$$(pr_\mathfrak{a}(\mathfrak{z}(\mathfrak{d}) \cap M_0))^{\perp_{\mathfrak{a}^\mathfrak{l}}} = \alpha_0(\ker [\cdot\,, \cdot]_\mathfrak{l}) \ .$$

This equation implies the equivalence of (b_0) and (B_0) and of (b_0') and (B_0').

Next we discuss condition (a_k) for $k \geq 1$. We consider the projection of $\mathfrak{d}$-modules $p_k : \mathfrak{d}_k \to R_k(\mathfrak{l})$. Then (a_k) is equivalent to $p_k(S(\mathfrak{d}_k)) = 0$. It turns out to be useful to express this condition in the following awkward way: Any $\mathfrak{d}$-submodule $\mathfrak{k} \subset p_k(S(\mathfrak{d}_k))$ vanishes.

We have $p_k(S(\mathfrak{d}_k)) \subset S(\mathfrak{l}) \cap R_k(\mathfrak{l})$. We claim that a $\mathfrak{d}$-submodule $\mathfrak{k} \subset S(\mathfrak{l}) \cap R_k(\mathfrak{l})$ is contained in $p_k(S(\mathfrak{d}_k))$ if and only if there exists an element $\Phi \in \mathrm{Hom}_\mathfrak{d}(\mathfrak{k}, \mathfrak{d}_k)$ such that $p_k \circ \Phi = \mathrm{Id}$. Indeed, if $\mathfrak{k} \subset p_k(S(\mathfrak{d}_k))$, then we can choose a $\mathfrak{d}$-invariant complement $\tilde{\mathfrak{k}}$ of $p_k^{-1}(0) \cap S(\mathfrak{d}_k)$ in the semi-simple $\mathfrak{d}$-module $p_k^{-1}(\mathfrak{k}) \cap S(\mathfrak{d}_k)$. The projection p_k maps $\tilde{\mathfrak{k}}$ isomorphically to $\mathfrak{k}$. Then we can take $\Phi = (p_k|_{\tilde{\mathfrak{k}}})^{-1}$. Vice versa, if Φ as above exists, then by semi-simplicity of $\mathfrak{k}$ the module $\tilde{\mathfrak{k}} := \Phi(\mathfrak{k})$ is semi-simple and therefore $p_k(S(\mathfrak{d}_k)) \supset p_k(\tilde{\mathfrak{k}}) = \mathfrak{k}$.

Let $i : \mathfrak{k} \to R_k(\mathfrak{l})$ be the natural inclusion. Observe that $[R(\mathfrak{l}), S(\mathfrak{l})] = 0$ implies $\mathfrak{k} \subset R_k(\mathfrak{l})^{R(\mathfrak{l})} \subset \mathfrak{z}(R_k(\mathfrak{l}))$. Using this fact and again the formulas (16) to (21) one shows that a homomorphism

$$\Phi = (\Phi_2, -\Phi_1, i) \in \mathrm{Hom}(\mathfrak{k}, R_k(\mathfrak{l})^*) \oplus \mathrm{Hom}(\mathfrak{k}, \mathfrak{a}) \oplus \mathrm{Hom}(\mathfrak{k}, R_k(\mathfrak{l})) = \mathrm{Hom}(\mathfrak{k}, \mathfrak{d}_k)$$

is $\mathfrak{d}$-equivariant if and only if the equations $(i),(ii)$ of condition (A_k) are satisfied and $\operatorname{im}\Phi_1 \in \mathfrak{a}^{R_k(\mathfrak{l})}$. The latter condition is vacuous since for $k \geq 1$ we have by semi-simplicity that $\rho|_{R_k(\mathfrak{l})} = 0$. We conclude that (a_k) is equivalent to (A_k).

Concerning (b_k), $k \geq 1$, the same reasoning as above yields that a $\mathfrak{d}$-submodule $\mathfrak{b} \subset \mathfrak{a}$ is contained in $pr_{\mathfrak{a}}(S(M_k))$ if and only if there exists an element $\Psi \in \operatorname{Hom}_{\mathfrak{d}}(\mathfrak{b}, M_k)$ such that $pr_{\mathfrak{a}} \circ \Psi = \operatorname{Id}$. An element $\Psi = (\Psi_1, i) \in \operatorname{Hom}(\mathfrak{b}, R_k(\mathfrak{l})^*) \oplus \operatorname{Hom}(\mathfrak{b}, \mathfrak{a}) = \operatorname{Hom}(\mathfrak{b}, M_k)$ is $\mathfrak{d}$-equivariant if and only if for all $L \in \mathfrak{l}$ and $B \in \mathfrak{b}$

$$\Psi_1(\rho(L)B) = -\langle B, \alpha(L,\cdot)\rangle_{\mathfrak{a}} - \langle \Psi_1(B), [L,\cdot]_{\mathfrak{l}}\rangle \in R_k(\mathfrak{l})^* \ .$$

If $\Phi = (\Psi_1 \circ i)^* \in \operatorname{Hom}(R_k(\mathfrak{l}), \mathfrak{a})$, then this equation is equivalent to

$$\langle \alpha(L,K), B\rangle_{\mathfrak{a}} = \langle \rho(L)\Phi(K) - \Phi([L,K]_{\mathfrak{l}}), B\rangle_{\mathfrak{a}} \ , \quad L \in \mathfrak{l}, K \in R_k(\mathfrak{l}), B \in \mathfrak{b}.$$

It follows that (b_k) is equivalent to (B_k) for all $k \geq 1$. $\square$

Remark 4.3 Theorem 4.1 shows in particular that in the case of abelian $\mathfrak{l}$ which has been studied in [KO 02] the extension $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$ is regularly balanced if and only if it is regular in the sense of [KO 02] and the representation ρ is semi-simple.

Each of the conditions $(a_k),(b_k)$, and (b_0') only depends on the equivalence class of the quadratic extension $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$. By Proposition 3.3 this implies that each of the conditions $(A_k),(B_k)$, and (B_0') only depends on the cohomology class $[\alpha,\gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$. It is an interesting exercise to check this directly.

Definition 4.5 *Let $\mathfrak{a}$ be a semi-simple orthogonal $\mathfrak{l}$-module. Let $m \in \mathbb{N}_0$ be such that $R_{m+1}(\mathfrak{l}) = 0$. A cohomology class $[\alpha,\gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ is called admissible if $(\alpha,\gamma) \in \mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$ satisfies the conditions (A_k), (B_k) for all $0 \leq k \leq m$. We denote the subset of admissible classes in $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ by $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp$. If $\mathfrak{a}$ is not semi-simple, then we set $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp = \emptyset$. Furthermore, a Lie algebra $\mathfrak{l}$ is called admissible if there exists a semi-simple orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ such that $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp \neq \emptyset$.*

Now we can reformulate Theorem 4.1 in the following way:

Corollary 4.2 *A quadratic extension is balanced if and only if the quadratic cohomology class assigned to it by (28) and (29) belongs to $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp$.*

5 Isomorphy and decomposability of metric Lie algebras

By the results of the previous sections (in particular Proposition 4.1, Theorem 3.1, and Corollary 4.2) the metric Lie algebras $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$ associated with semi-simple orthogonal modules $(\rho,\mathfrak{a})$ of a Lie algebra $\mathfrak{l}$ and $[\alpha,\gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp$ exhaust all isomorphism classes of metric Lie algebras without simple ideals. In order to approach the classification of indecomposable metric Lie algebras we have to decide which of these data

lead to isomorphic or decomposable metric Lie algebras, respectively. This is the first aim of the present section. We conclude the section giving a classification scheme for isomorphism classes of non-simple indecomposable metric Lie algebras.

Recall the definition of morphisms $(S,U):(\mathfrak{l}_1,\mathfrak{a}_1)\to(\mathfrak{l}_2,\mathfrak{a}_2)$ of pairs (of Lie algebras and orthogonal modules) from Section 2 around Equation (7). If (S,U) is an isomorphism of pairs and $(\mathfrak{g},\mathfrak{i},i,p)$ is a quadratic extension of $\mathfrak{l}_1$ by $\mathfrak{a}_1$, then we observe that $(\mathfrak{g},\mathfrak{i},i\circ U,S\circ p)$ is a quadratic extension of $\mathfrak{l}_2$ by $\mathfrak{a}_2$.

Lemma 5.1 *Let $(\rho_j,\mathfrak{a}_j)$, $j=1,2$, be orthogonal modules of Lie algebras $\mathfrak{l}_j$, and let $(\mathfrak{g}_j,\mathfrak{i}_j,i_j,p_j)$ be quadratic extensions of $\mathfrak{l}_j$ by $\mathfrak{a}_j$.*

(a) *If there is an isomorphism of pairs $(S,U):(\mathfrak{l}_1,\mathfrak{a}_1)\to(\mathfrak{l}_2,\mathfrak{a}_2)$ such that the quadratic extensions $(\mathfrak{g}_1,\mathfrak{i}_1,i_1\circ U,S\circ p_1)$ and $(\mathfrak{g}_2,\mathfrak{i}_2,i_2,p_2)$ are equivalent, then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic as metric Lie algebras.*

(b) *If the quadratic extensions $(\mathfrak{g}_j,\mathfrak{i}_j,i_j,p_j)$, $i=1,2$, are balanced, and the metric Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic, then there exists an isomorphism of pairs $(S,U):(\mathfrak{l}_1,\mathfrak{a}_1)\to(\mathfrak{l}_2,\mathfrak{a}_2)$ such that the extensions $(\mathfrak{g}_1,\mathfrak{i}_1,i_1\circ U,S\circ p_1)$ and $(\mathfrak{g}_2,\mathfrak{i}_2,i_2,p_2)$ are equivalent.*

Proof. Part (a) is a triviality since any equivalence of quadratic extensions is by definition an isomorphism of metric Lie algebras. Let us prove (b). Let $F:\mathfrak{g}_1\to\mathfrak{g}_2$ be an isomorphism of metric Lie algebras. Then $F(\mathfrak{i}(\mathfrak{g}_1))=\mathfrak{i}(\mathfrak{g}_2)$ and $F(\mathfrak{j}(\mathfrak{g}_1))=\mathfrak{j}(\mathfrak{g}_2)$. The quadratic extensions are balanced, i.e., $\mathfrak{i}_j=\mathfrak{i}(\mathfrak{g}_j)$. Thus F induces a Lie algebra isomorphism

$$\bar F:\mathfrak{g}_1/\mathfrak{i}_1\longrightarrow\mathfrak{g}_2/\mathfrak{i}_2$$

such that $\bar F(\ker p_1)=\ker p_2$. We then define (S,U) by

$$S(p_1(X))=p_2(\bar F(X))\ ,\ X\in\mathfrak{g}_1/\mathfrak{i}_1,\quad i_1(U(A))=\bar F^{-1}(i_2(A))\ ,\ A\in\mathfrak{a}_2\ .$$

We compute for $L\in\mathfrak{l}_1$, and $\tilde L\in\mathfrak{g}_1/\mathfrak{i}$ such that $p_1(\tilde L)=L$ and $A\in\mathfrak{a}_2$

$$\begin{aligned} i_1\circ\rho_1(L)\circ U(A) &= [\tilde L,i_1\circ U(A)]=\bar F^{-1}[\bar F(\tilde L),i_2(A)] \\ &= \bar F^{-1}\circ i_2\circ\rho_2(S(L))(A)=i_1\circ U\circ\rho_2(S(L))(A)\ . \end{aligned}\tag{45}$$

In the third step we have used that $p_2(\bar F(\tilde L))=S(L)$. Equation (45) shows that (S,U) is an isomorphism of pairs. Now F defines an equivalence between $(\mathfrak{g}_1,\mathfrak{i}_1,i_1\circ U,S\circ p_1)$ and $(\mathfrak{g}_2,\mathfrak{i}_2,i_2,p_2)$. □

We can now give a necessary and sufficient criterion of isomorphy of metric Lie algebras in terms of the admissible quadratic cohomology classes associated with their structures as a balanced quadratic extensions (see Proposition 4.1).

Proposition 5.1 *Let $(\rho_i,\mathfrak{a}_i)$, $i=1,2$, be orthogonal modules of Lie algebras $\mathfrak{l}_i$, and let $(\alpha_i,\gamma_i)\in\mathcal{Z}^2_Q(\mathfrak{l}_i,\mathfrak{a}_i)$.*

(a) *If there exists an isomorphism of pairs* $(S,U) : (\mathfrak{l}_1, \mathfrak{a}_1) \to (\mathfrak{l}_2, \mathfrak{a}_2)$ *such that* $(S,U)^*[\alpha_2,\gamma_2] = [\alpha_1,\gamma_1] \in \mathcal{H}^2_Q(\mathfrak{l}_1,\mathfrak{a}_1)$*, then* $\mathfrak{d}_{\alpha_1,\gamma_1}(\mathfrak{l}_1,\mathfrak{a}_1,\rho_1)$ *and* $\mathfrak{d}_{\alpha_2,\gamma_2}(\mathfrak{l}_2,\mathfrak{a}_2,\rho_2)$ *are isomorphic as metric Lie algebras.*

(b) *If* $[\alpha_i,\gamma_i] \in \mathcal{H}^2_Q(\mathfrak{l}_i,\mathfrak{a}_i)_\sharp$*,* $i = 1,2$*, and* $\mathfrak{d}_{\alpha_1,\gamma_1}(\mathfrak{l}_1,\mathfrak{a}_1,\rho_1)$ *and* $\mathfrak{d}_{\alpha_2,\gamma_2}(\mathfrak{l}_2,\mathfrak{a}_2,\rho_2)$ *are isomorphic metric Lie algebras, then there is an isomorphism of pairs* $(S,U) : (\mathfrak{l}_1,\mathfrak{a}_1) \to (\mathfrak{l}_2,\mathfrak{a}_2)$ *such that* $(S,U)^*[\alpha_2,\gamma_2] = [\alpha_1,\gamma_1] \in \mathcal{H}^2_Q(\mathfrak{l}_1,\mathfrak{a}_1)_\sharp$*.*

Proof. We write the quadratic extensions $\mathfrak{d}_{\alpha_j,\gamma_j}(\mathfrak{l}_j,\mathfrak{a}_j,\rho_j)$ as $(\mathfrak{d}_j,\mathfrak{l}_j^*,i_j,p_j)$. Let $s : \mathfrak{l}_1 \to \mathfrak{d}_1$ be the embedding. Let $(S,U) : (\mathfrak{l}_1,\mathfrak{a}_1) \to (\mathfrak{l}_2,\mathfrak{a}_2)$ be an isomorphism of pairs. Then $\tilde{s} := s \circ S^{-1} : \mathfrak{l}_2 \to \mathfrak{d}_1$ is a section of $S \circ \tilde{p}_1$ with isotropic image. Now the cocycle associated with the quadratic extension $(\mathfrak{d}_1,\mathfrak{l}_1^*,i_1\circ U, S\circ p_1)$ of $\mathfrak{l}_2$ by $\mathfrak{a}_2$ and the section $\tilde{s}$ is given by

$$\left((S^{-1},U^{-1})^*\alpha_1,(S^{-1})^*\gamma_1\right)$$

(see (28) and (29)). Therefore the proposition is a consequence of Lemma 5.1, Proposition 3.3, and Corollary 4.2. □

There is a natural notion of a direct sum of quadratic extensions. Namely, if $(\mathfrak{g}_j,\mathfrak{i}_j,i_j,p_j)$, $j = 1,2$, are quadratic extensions of Lie algebras $\mathfrak{l}_j$ by orthogonal modules $\mathfrak{a}_j$, then

$$(\mathfrak{g}_1\oplus\mathfrak{g}_2,\mathfrak{i}_1\oplus\mathfrak{i}_2,i_1\oplus i_2,p_1\oplus p_2)$$

is a quadratic extension of $\mathfrak{l}_1\oplus\mathfrak{l}_2$ by $\mathfrak{a}_1\oplus\mathfrak{a}_2$. A quadratic extension is called decomposable, if it can be written as a non-trivial direct sum of two quadratic extensions. If a quadratic extension is equivalent to a decomposable one, then it is decomposable. Of course, the decomposability of a quadratic extension $(\mathfrak{g},\mathfrak{i},i,p)$ implies the decomposability of $\mathfrak{g}$ as a metric Lie algebra. The opposite assertion is not true in general. However, we have

Lemma 5.2 *Let* $(\mathfrak{g},\mathfrak{i},i,p)$ *be a balanced quadratic extension of* $\mathfrak{l}$ *by* $\mathfrak{a}$*. If the metric Lie algebra* $\mathfrak{g}$ *is decomposable, then the quadratic extension* $(\mathfrak{g},\mathfrak{i},i,p)$ *is decomposable, too.*

Proof. If $\mathfrak{g} = \mathfrak{g}_1\oplus\mathfrak{g}_2$ and $(\mathfrak{g},\mathfrak{i},i,p)$ is balanced, then $\mathfrak{i} = \mathfrak{i}(\mathfrak{g}) = \mathfrak{i}(\mathfrak{g}_1)\oplus\mathfrak{i}(\mathfrak{g}_2)$. In particular, $\mathfrak{g}/\mathfrak{i} = \mathfrak{g}_1/\mathfrak{i}(\mathfrak{g}_1)\oplus\mathfrak{g}_2/\mathfrak{i}(\mathfrak{g}_2)$. We set $\mathfrak{l}_j := p(\mathfrak{g}_j/\mathfrak{i}(\mathfrak{g}_j))$, $p_j := p|_{\mathfrak{g}_j/\mathfrak{i}(\mathfrak{g}_j)}$, $\mathfrak{a}_j = i^{-1}(\mathfrak{g}_j/\mathfrak{i}(\mathfrak{g}_j))$, $i_j := i|_{\mathfrak{a}_j}$. Then $(\mathfrak{g}_j,\mathfrak{i}_j,i_j,p_j)$ are quadratic extensions of $\mathfrak{l}_j$ by $\mathfrak{a}_j$, and

$$(\mathfrak{g},\mathfrak{i},i,p) = (\mathfrak{g}_1\oplus\mathfrak{g}_2,\mathfrak{i}(\mathfrak{g}_1)\oplus\mathfrak{i}(\mathfrak{g}_2),i_1\oplus i_2,p_1\oplus p_2)\ .$$

□

Recall the notion of a non-trivial direct sum of pairs from Defininition 2.3.

Definition 5.1 *Let* $(\rho,\mathfrak{a})$ *be an orthogonal module of a Lie algebra* $\mathfrak{l}$*. A cohomology class* $c\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ *is called decomposable if there is a decomposition*

$$(\mathfrak{l},\mathfrak{a}) = (\mathfrak{l}_1,\mathfrak{a}_1)\oplus(\mathfrak{l}_2,\mathfrak{a}_2)$$

into a non-trivial direct sum of pairs such that in the notation of Lemma 2.2

$$c \in (q_1, j_1)^* \mathcal{H}^2_Q(\mathfrak{l}_1, \mathfrak{a}_1) + (q_2, j_2)^* \mathcal{H}^2_Q(\mathfrak{l}_2, \mathfrak{a}_2) \ .$$

A cohomology class which is not decomposable is called indecomposable. We denote the set of all indecomposable elements in $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_\sharp$ by $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0$.

Proposition 5.2 *Let $(\rho, \mathfrak{a})$ be an orthogonal module of a Lie algebra $\mathfrak{l}$, and let $(\alpha, \gamma) \in \mathcal{Z}^2_Q(\mathfrak{l}, \mathfrak{a})$.*

(a) *If $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ is decomposable, then the quadratic extension $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}, \rho)$ is decomposable.*

(b) *If $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}_i, \mathfrak{a}_i)_\sharp$ and $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}, \rho)$ is decomposable as a metric Lie algebra, then $[\alpha, \gamma]$ is decomposable.*

Proof. Let $\mathfrak{l}_i$, $\mathfrak{a}_i$, j_i, and q_i be as in Definition 5.1. We first note that, if a quadratic extension $(\mathfrak{g}, \mathfrak{i}, i, p)$ is the direct sum

$$(\mathfrak{g}_1, \mathfrak{i}_1, i_1, p_1) \oplus (\mathfrak{g}_2, \mathfrak{i}_2, i_2, p_2)$$

of quadratic extensions of $\mathfrak{l}_i$ by $\mathfrak{a}_i$ with associated cohomology classes $c_i \in \mathcal{H}^2_Q(\mathfrak{l}_i, \mathfrak{a}_i)$, then the cohomology class associated with $(\mathfrak{g}, \mathfrak{i}, i, p)$ is given by $(q_1, j_1)^* c_1 + (q_2, j_2)^* c_2 \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$.

Assume now that $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ is decomposable. Then there exist elements $(\alpha_i, \gamma_i) \in \mathcal{Z}^2_Q(\mathfrak{l}_i, \mathfrak{a}_i)$ such that $[\alpha, \gamma] = (q_1, j_1)^*[\alpha_1, \gamma_1] + (q_2, j_2)^*[\alpha_2, \gamma_2]$. By the above and Theorem 3.1 the quadratic extension $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}, \rho)$ is equivalent to the direct sum $\mathfrak{d}_{\alpha_1,\gamma_1}(\mathfrak{l}_1, \mathfrak{a}_1, \rho_1) \oplus \mathfrak{d}_{\alpha_2,\gamma_2}(\mathfrak{l}_2, \mathfrak{a}_2, \rho_2)$ and therefore decomposable. This proves (a).

If $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}_i, \mathfrak{a}_i)_\sharp$ and $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}, \rho)$ is decomposable as a metric Lie algebra, then $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}, \rho)$ is balanced and thus by Lemma 5.2 decomposable as a quadratic extension. Then the discussion at the beginning of the proof shows that $[\alpha, \gamma]$ is decomposable. □

We conclude this section with a classification scheme for isomorphism classes of non-simple indecomposable metric Lie algebras.

Let us fix a Lie algebra $\mathfrak{l}$ and a semi-Euclidean vector space $(\mathfrak{a}, \langle \cdot, \cdot \rangle_\mathfrak{a})$. We consider the set $\mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(\mathfrak{a}, \langle \cdot, \cdot \rangle_\mathfrak{a}))_{\mathrm{ss}}$ of all orthogonal semi-simple representations of $\mathfrak{l}$ on $\mathfrak{a}$. If $\rho \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(\mathfrak{a}, \langle \cdot, \cdot \rangle_\mathfrak{a}))_{\mathrm{ss}}$ is fixed we denote the corresponding $\mathfrak{l}$-module by $\mathfrak{a}_\rho$. The group $G := \mathrm{Aut}(\mathfrak{l}) \times O(\mathfrak{a}, \langle \cdot, \cdot \rangle_\mathfrak{a})$ acts from the right on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(\mathfrak{a}, \langle \cdot, \cdot \rangle_\mathfrak{a}))_{\mathrm{ss}}$ by

$$(S, U)^* \rho := \mathrm{Ad}(U^{-1}) \circ S^* \rho \ , \qquad S \in \mathrm{Aut}(\mathfrak{l}),\ U \in O(\mathfrak{a}, \langle \cdot, \cdot \rangle_\mathfrak{a}) \ .$$

Then for any $\rho \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(\mathfrak{a}, \langle \cdot, \cdot \rangle_\mathfrak{a}))_{\mathrm{ss}}$ an element $g = (S, U) \in G$ defines an isomorphism of pairs $\bar{g} := (S, U^{-1}) : (\mathfrak{l}, \mathfrak{a}_{g^*\rho}) \to (\mathfrak{l}, \mathfrak{a}_\rho)$ and therefore induces a bijection

$$\bar{g}^* : \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}_\rho) \to \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}_{g^*\rho}) \ .$$

We obtain a right action of G on the disjoint union

$$\coprod_{\rho\in\mathrm{Hom}(\mathfrak{l},\mathfrak{so}(\mathfrak{a},\langle\cdot\,,\cdot\rangle_{\mathfrak{a}}))_{\mathrm{ss}}} \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)\ .$$

If it is clear from the context that a pair $g = (S, U)$ is considered as an element of G, then its action on the above space will be simply denoted by g^* or $(S, U)^*$. Note the slight difference of the meaning of $(S, U)^*$, if (S, U) is a morphism of pairs.

As in Definition 5.1 let $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_0 \subset \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)$ be the subset of all admissible indecomposable elements (see Definition 4.5 and Theorem 4.1 for the admissibility conditions). Then the set

$$\coprod_{\rho\in\mathrm{Hom}(\mathfrak{l},\mathfrak{so}(\mathfrak{a},\langle\cdot\,,\cdot\rangle_{\mathfrak{a}}))_{\mathrm{ss}}} \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_0$$

is G-invariant. Combining Proposition 4.1 and Corollary 4.2 with Propositions 5.1 and 5.2 we obtain

Theorem 5.1 *Let $\mathfrak{l}$ be a Lie algebra, and let $(\mathfrak{a},\langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ be a semi-Euclidean vector space. We consider the class $\mathcal{A}(\mathfrak{l},\mathfrak{a})$ of non-simple indecomposable metric Lie algebras $\mathfrak{g}$ satisfying*

1. *The Lie algebras $\mathfrak{g}/\mathfrak{j}(\mathfrak{g})$ and $\mathfrak{l}$ are isomorphic.*
2. *$\mathfrak{j}(\mathfrak{g})/\mathfrak{i}(\mathfrak{g})$ is isomorphic to $(\mathfrak{a},\langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ as a semi-Euclidean vector space.*

Then the set of isomorphism classes of $\mathcal{A}(\mathfrak{l},\mathfrak{a})$ is in bijective correspondence with the orbit space of the action of $G = \mathrm{Aut}(\mathfrak{l}) \times O(\mathfrak{a},\langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ on

$$\coprod_{\rho\in\mathrm{Hom}(\mathfrak{l},\mathfrak{so}(\mathfrak{a},\langle\cdot\,,\cdot\rangle_{\mathfrak{a}}))_{\mathrm{ss}}} \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_0\ .$$

This orbit space can also be written as

$$\coprod_{[\rho]\in\mathrm{Hom}(\mathfrak{l},\mathfrak{so}(\mathfrak{a},\langle\cdot\,,\cdot\rangle_{\mathfrak{a}}))_{\mathrm{ss}}/G} \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_0/G_\rho\ ,$$

where $G_\rho = \{g \in G \mid g^*\rho = \rho\}$ is the automorphism group of the pair $(\mathfrak{l},\mathfrak{a}_\rho)$.

In view of Proposition 2.2 the automorphism group G_ρ can be replaced by the group of outer automorphisms of $(\mathfrak{l},\mathfrak{a}_\rho)$.

Note that $\mathcal{A}(\mathfrak{l},\mathfrak{a})$ is empty if the Lie algebra $\mathfrak{l}$ is not admissible (see Definition 4.5). We will use Theorem 5.1 in order to provide a classification of all indecomposable metric Lie algebras of index 3 (see Theorem 7.1). In the way of this classification, in particular in Section 6, we shall see that there are many isomorphism classes of non-admissible Lie algebras $\mathfrak{l}$. In order to apply Theorem 5.1 to more general situations one would like to have (as a first step) a good description of the class $\mathcal{M}$ of all admissible Lie algebras. Up to now we only know that $\mathcal{M}$ contains all reductive Lie algebras as well as some solvable, nilpotent and mixed ones, and that $\mathcal{M}$ is closed under forming direct sums.

6 Solvable admissible Lie algebras with small radical

In this section we will classify all solvable admissible Lie algebras whose radical of nilpotency is one- or two-dimensional. Recall that a Lie algebra $\mathfrak{l}$ is called admissible if there exists a balanced quadratic extension $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$. On the one hand this classification serves as an example which shows how one can handle the admissibility conditions. On the other hand we will need the results in Section 7.1 in order to classify the metric Lie algebras of index 3.

6.1 Weight spaces

Throughout this section we will assume that $\mathfrak{l}$ is a solvable Lie algebra whose radical of nilpotency $R := R(\mathfrak{l}) = \mathfrak{l}'$ is abelian. Moreover, let $(\rho,\mathfrak{a})$ be a semi-simple orthogonal $\mathfrak{l}$-module. Since R is abelian

$$\mathrm{ad}_0 : \mathfrak{l} \longrightarrow \mathfrak{gl}(R), \quad L \longmapsto \mathrm{ad}(L)|_R$$

induces a representation of the abelian Lie algebra $\mathfrak{l}/R$ on R, which we also denote by ad_0. Furthermore, since ρ is semi-simple we have $\rho|_R = 0$ and $\mathfrak{a}$ can be considered as a semi-simple $\mathfrak{l}/R$-module. Hence the complexification $\mathfrak{a}_{\mathbb{C}}$ of $\mathfrak{a}$ decomposes into $\mathfrak{a}_{\mathbb{C}} = \bigoplus_{\lambda\in\Lambda} E_\lambda$, where

$$\Lambda = (\mathfrak{l}_{\mathbb{C}}/R_{\mathbb{C}})^* \cong \{\lambda \in \mathfrak{l}_{\mathbb{C}}^* \mid \lambda|_{R_{\mathbb{C}}} = 0\} \subset \mathfrak{l}_{\mathbb{C}}^*$$

and

$$E_\lambda = \{A \in \mathfrak{a}_{\mathbb{C}} \mid \rho(L)(A) = \lambda(L)\cdot A \text{ for all } L \in \mathfrak{l}\}.$$

Let $p_\lambda : \mathfrak{a}_{\mathbb{C}} \to E_\lambda$ denote the projection.

Let $\langle\cdot\,,\cdot\rangle$ on $\mathfrak{a}_{\mathbb{C}}$ be the sesquilinear extension of $\langle\cdot\,,\cdot\rangle$ on $\mathfrak{a}_{\mathbb{C}}$. Then $E_\lambda \perp E_\mu$ if not $\mu = -\bar{\lambda}$. Let us also define analogous spaces for the complexification $R_{\mathbb{C}}$ of ad_0 by

$$V_\lambda := \{U \in R_{\mathbb{C}} \mid \mathrm{ad}_0(L)(U) = \lambda(L)\cdot U \text{ for all } L \in \mathfrak{l}\}.$$

Lemma 6.1 *Let R be abelian and $\rho \in \mathrm{Hom}(\mathfrak{l},\mathfrak{so}(\mathfrak{a},\langle\cdot\,,\cdot\rangle_{\mathfrak{a}}))_{\mathrm{ss}}$. Assume $\alpha \in Z^2(\mathfrak{l},\mathfrak{a})$ satisfies $\alpha|_{R\times R} = 0$. Then there exists a cocycle $\tilde{\alpha} \in Z^2(\mathfrak{l},\mathfrak{a})$ such that $\tilde{\alpha}|_{R\times R} = 0$, $[\alpha] = [\tilde{\alpha}] \in H^2(\mathfrak{l},\mathfrak{a})$ and $\tilde{\alpha}(\mathfrak{l},V_\lambda) \subset E_\lambda$ holds for all $\lambda \in \Lambda$.*

Proof. Because of $\alpha|_{R\times R} = 0$ and $\rho|_R = 0$ the cocycle α defines a cocycle $\bar{\alpha} \in Z^1(\mathfrak{l}/R, C^1(R,\mathfrak{a}))$ by $\bar{\alpha}(L+R)(U) = \alpha(L,U)$ for $L \in \mathfrak{l},\ U \in R$. Since $\mathfrak{l}/R$ is abelian we have $H^1(\mathfrak{l}/R, C^1(R,\mathfrak{a})) = H^1(\mathfrak{l}/R, C^1(R,\mathfrak{a})^{(\mathfrak{l}/R)})$. Here $C^1(R,\mathfrak{a})^{(\mathfrak{l}/R)}$ denotes the nilsubspace of $C^1(R,\mathfrak{a})$ with respect to the action of $\mathfrak{l}/R$. Thus there exists a cochain $\bar{\tau} \in C^0(\mathfrak{l}/R, C^1(R,\mathfrak{a})) \cong C^1(R,\mathfrak{a})$ such that $\bar{\alpha} + d\bar{\tau}$ has values only in $C^1(R,\mathfrak{a})^{(\mathfrak{l}/R)}$. Hence for all $L' \in \mathfrak{l}$ there exists a $k \in \mathbb{N}$ such that $(L')^k \cdot ((\bar{\alpha}+d\bar{\tau})(L+R)) = 0$. For $U \in V_\lambda$ we have

$$0 = (\,(L')^k \cdot ((\bar{\alpha}+d\bar{\tau})(L+R))\,)(U) = (\rho(L') - \lambda(L'))^k(\,(\bar{\alpha}+d\bar{\tau})(L+R)(U)\,).$$

Since ρ is semi-simple we get $(\bar{\alpha}+d\bar{\tau})(L+R)(U) \in E_\lambda$ for all $L \in \mathfrak{l}$. Now we can choose a linear extension $\tau \in C^1(\mathfrak{l}, \mathfrak{a})$ of $\bar{\tau}$ and obtain $(\alpha+d\tau)(L,U) = (\bar{\alpha}+d\bar{\tau})(L+R)(U) \in E_\lambda$. $\square$

6.2 $\dim R(\mathfrak{l}) = 1$

In the following we will often describe a Lie algebra $\mathfrak{l}$ giving only the non-trivial Lie bracket relations between vectors of a basis. We will use this notation only for those Lie algebras for which all basis vectors appear in one of these relations.

In Section 7.1 we will see that the Heisenberg algebra

$$\mathfrak{h}(1) = \{[X,Y] = Z\}$$

is admissible. On the other hand we can prove the following.

Proposition 6.1 *If $\mathfrak{l}$ is a solvable admissible Lie algebra with $\dim R(\mathfrak{l}) = 1$, then*

$$\mathfrak{l} \cong \mathfrak{h}(1) \oplus \mathbb{R}^k.$$

Proof. We choose $Z \in \mathfrak{l}$ such that $R = \mathbb{R} \cdot Z$. Then we define $\lambda \in \Lambda$ by $[L,Z] = \lambda(L)Z$. Obviously, λ is real. By Lemma 6.1 we may assume $\alpha(Z,\mathfrak{l}) \subset E_\lambda$. Suppose $\lambda \neq 0$. Then E_λ is isotropic and therefore $E_\lambda \subset \mathfrak{b}_1$ (see Theorem 4.1 (B_1), $\Phi = 0$ is a solution). By (B_1) the space $\mathfrak{b}_1$ is non-degenerate. Therefore also $E_{-\lambda} \subset \mathfrak{b}_1$. By definition of $\mathfrak{b}_1$ there exists an element $\Phi \in \mathrm{Hom}(R,\mathfrak{a})$ such that

$$\langle \alpha(L,Z), B\rangle = \langle \rho(L)(\Phi(Z)) - \Phi([L,Z]), B\rangle = \langle (\rho(L) - \lambda(L))(\Phi(Z)), B\rangle = 0$$

for all $B \in E_{-\lambda}$. Hence $\alpha(L,Z) = 0$ for all $L \in \mathfrak{l}$. Condition (A_1) now implies $R = 0$ which is a contradiction. Thus $\lambda = 0$. This implies

$$\mathfrak{l} \cong \{[X_{2i-1}, X_{2i}] = Z \mid i = 1, ..., r\} \oplus \mathbb{R}^k.$$

For $r > 1$ the cocycle condition on α yields $\alpha(Z,\cdot) = 0$. Hence $r = 1$ and the assertion follows. $\square$

6.3 $\dim R(\mathfrak{l}) = 2$

We consider the following Lie algebras with two-dimensional radical of nilpotency:

$$\begin{aligned} \mathfrak{n}(2) &= \{[X,Y] = Z,\ [X,Z] = -Y\}, \\ \mathfrak{r}_{3,-1} &= \{[X,Y] = Y,\ [X,Z] = -Z\}, \\ \mathfrak{r}_{3,-2} &= \{[X,Y] = -2Y,\ [X,Z] = Z\}. \end{aligned}$$

All these Lie algebras are admissible. For $\mathfrak{n}(2)$ and $\mathfrak{r}_{3,-1}$ we will see this in Section 7.1. As for $\mathfrak{r}_{3,-2}$ consider $\mathfrak{a} = \mathbb{R}^{1,1}$ spanned by the isotropic vectors e_+ and e_-. Let ρ be given by

$$\rho(X)(e_+) = e_+,\ \rho(X)(e_-) = -e_-,\ \rho(Y) = \rho(Z) = 0.$$

We define $(\alpha,\gamma)\in\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)$ by

$$\alpha(Y,Z)=e_-,\ \alpha(X,Z)=e_+,\ \alpha(X,Y)=0,\ \gamma=0.$$

Then $[\alpha,0]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_\sharp$, hence $\mathfrak{r}_{3,-2}$ is admissible.

In this section we will prove the following classification result.

Proposition 6.2 *If $\mathfrak{l}$ is a solvable, non-nilpotent admissible Lie algebra with $\dim R(\mathfrak{l})=2$, then $\mathfrak{l}$ is isomorphic to one of the Lie algebras*

$$\mathfrak{n}(2)\oplus\mathbb{R}^k,\ \ \mathfrak{r}_{3,-1}\oplus\mathbb{R}^k,\ \ \mathfrak{r}_{3,-2}\oplus\mathbb{R}^k.$$

In the following let $\mathfrak{l}$ be a solvable, non-nilpotent Lie algebra with $\dim R=2$. Since in this case R is (two-dimensional and) nilpotent it must be abelian.

Let us first assume that the representation ad_0 of $\mathfrak{l}_\mathbb{C}/R_\mathbb{C}$ on $R_\mathbb{C}$ is semi-simple. Then ad_0 has two weights $\lambda^1,\ \lambda^2$ and either λ^1 and λ^2 are real weights or $\lambda^1=\overline{\lambda^2}$ are complex weights. In both cases we may assume $\lambda^1\ne0$ since $\mathfrak{l}$ is not nilpotent.

Lemma 6.2 *Any cocycle $\alpha\in Z^2(\mathfrak{l},\mathfrak{a})$ satisfies $\alpha(R,R)\subset E_{\lambda^1+\lambda^2}$. There exists a cocycle $\tilde\alpha\in Z^2(\mathfrak{l},\mathfrak{a})$ such that $[\alpha]=[\tilde\alpha]\in H^2(\mathfrak{l},\mathfrak{a})$ and $\tilde\alpha(R,\mathfrak{l})\subset E_{\lambda^1+\lambda^2}+E_{\lambda^1}+E_{\lambda^2}$.*

Proof. The first assertion follows from the cocycle condition. To prove the second one we look at the decomposition $\alpha=p_{\lambda^1+\lambda^2}\circ\alpha+\alpha'$. Then the first assertion implies $\alpha'|_{R\times R}=0$. Hence we may apply Lemma 6.1 to α' and the second assertion follows. □

Lemma 6.3 *Let $\mathfrak{l}$ be admissible and $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp$. Assume $\lambda^1+\lambda^2\ne0$. Let $\mathfrak{k}\subset R(\mathfrak{l})\cap S(\mathfrak{l})=R(\mathfrak{l})$ be an $\mathfrak{l}$-ideal. If $\alpha(K,L)=0$ for all $K\in\mathfrak{k}$ and $L\in\mathfrak{l}$, then $\mathfrak{k}=0$.*

Proof. If $\alpha(K,L)=0$ for all $K\in\mathfrak{k}$ and $L\in\mathfrak{l}$, then $(A_1)\,(i)$ is satisfied for $\Phi_1=0$. We will show that $(A_1)\,(ii)$ is also satisfied. Suppose $L_1,L_2\in\mathfrak{l}$, $K\in\mathfrak{k}$ and $U\in R$. Because of $\alpha(K,\cdot)=0$ we have $d\gamma(K,U,L_1,L_2)=\frac12\langle\alpha\wedge\alpha\rangle(K,U,L_1,L_2)=0$. On the other hand $d\gamma(K,U,L_1,L_2)=-(\lambda^1+\lambda^2)(L_1)\gamma(K,U,L_2)+(\lambda^1+\lambda^2)(L_2)\gamma(K,U,L_1)$ holds where we used $\dim R=2$. Hence

$$(\lambda^1+\lambda^2)(L_1)\gamma(K,U,L_2)=(\lambda^1+\lambda^2)(L_2)\gamma(K,U,L_1)\,. \tag{46}$$

Now let $L_0\in\mathfrak{l}$ be such that $(\lambda^1+\lambda^2)(L_0)\ne0$ and define $\Phi_2\in\mathrm{Hom}(\mathfrak{k},R(\mathfrak{l})^*)$ by

$$\langle\Phi_2(K),U\rangle=\tfrac{1}{(\lambda^1+\lambda^2)(L_0)}\gamma(L_0,K,U)\,.$$

Then we have

$$\begin{aligned}\langle\Phi_2(K),[L,U]\rangle+\langle\Phi_2([L,K]),U\rangle &= \tfrac{1}{(\lambda^1+\lambda^2)(L_0)}\Big(\gamma(L_0,K,[L,U])+\gamma(L_0,[L,K],U)\Big)\\ &= \tfrac{(\lambda^1+\lambda^2)(L)}{(\lambda^1+\lambda^2)(L_0)}\gamma(L_0,K,U)\ =\ \gamma(L,K,U)\end{aligned}$$

by (46). Hence $(A_1)\,(ii)$ is satisfied for Φ_1, Φ_2 as above. Consequently $\mathfrak{k}=0$. □

Lemma 6.4 *If $\mathfrak{l}$ is admissible, then $\lambda^1+\lambda^2\in\{0,-\lambda^1,-\lambda^2\}$.*

Proof. Since $\mathfrak{l}$ is admissible we can choose a cohomology class $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp$ for a suitable orthogonal $\mathfrak{l}$-module $\mathfrak{a}$. By Lemma 6.2 we may suppose $\alpha(\mathfrak{l},R)\subset E_{\lambda^1}+E_{\lambda^2}+E_{\lambda^1+\lambda^2}$. As above we assume $\lambda^1\neq 0$ since $\mathfrak{l}$ is not nilpotent. Assume now $\lambda^1+\lambda^2\notin\{0,-\lambda^1,-\lambda^2\}$. Then we have

$$\{\lambda^1,\overline{\lambda^1},\lambda^1+\lambda^2\}\ \cap\ \{-\overline{\lambda^1},-\overline{\lambda^2},-(\lambda^1+\lambda^2)\}=\emptyset \tag{47}$$

$$\{\lambda^1,\overline{\lambda^1},\lambda^1+\lambda^2\}\ \cap\ \{-\overline{\lambda^1},-\lambda^1,-(\lambda^1+\lambda^2)\}=\emptyset\quad . \tag{48}$$

From (47) we obtain

$$E_{\mathbb{C}}:=E_{\lambda^1}+E_{\overline{\lambda^1}}+E_{\lambda^1+\lambda^2}\perp E_{\lambda^1}+E_{\lambda^2}+E_{\lambda^1+\lambda^2}.$$

The vector space $E_{\mathbb{C}}$ is invariant under conjugation, thus it is the complex span of a real subspace $E\subset\mathfrak{a}$. By (48) E is totally isotropic. We have $E\subset\mathfrak{b}_1$ since $\alpha(\mathfrak{l},R)\subset E_{\lambda^1}+E_{\lambda^2}+E_{\lambda^1+\lambda^2}$ implies $\langle\alpha(\mathfrak{l},R),B\rangle=0$ for all $B\in E$, hence $\Phi=0$ is a solution. By (B_1) the space $\mathfrak{b}_1$ is non-degenerate, thus we have

$$E':=\mathfrak{a}\cap(\,E_{-\overline{\lambda^1}}+E_{-\lambda^1}+E_{-(\lambda^1+\lambda^2)}\,)\subset\mathfrak{b}_1.$$

Therefore there exists a homomorphism $\Phi\in\mathrm{Hom}(R,\mathfrak{a})$ such that

$$\langle\alpha(L,U),B\rangle=\langle\rho(L)\Phi(U)-\Phi([L,U]),B\rangle=0$$

for all $L\in\mathfrak{l}$, $U\in R$ and $B\in E'$. In particular, we have $\langle\alpha(Y,Z),B\rangle=\langle\rho(Y)\Phi(Z)-\Phi([Y,Z]),B\rangle=0$ for all $Y,Z\in R$ and $B\in E_{-(\lambda^1+\lambda^2)}\subset E'$. Since $\alpha|_{R\times R}\subset E_{\lambda^1+\lambda^2}$ this implies $\alpha|_{R\times R}=0$. According to Lemma 6.3 there are elements $U_1\in V_{\lambda^1}$ and $L\in\mathfrak{l}$ such that $\alpha(U_1,L)\neq 0$. By Lemma 6.1 we may assume $\alpha(U_1,L)\in E_{\lambda^1}$. Hence there exists an element $B\in E_{-\overline{\lambda^1}}$ such that $\langle\alpha(L,U_1),B\rangle\neq 0$. Then

$$\langle\alpha(L,U_{1i}),B_j\rangle=\langle\rho(L)\Phi(U_{1i})-\Phi([L,U_{1i}],B_j\rangle,\quad i,j\in\{1,2\}$$

for $U_{11}=\mathrm{Re}\,U_1$, $U_{12}=\mathrm{Im}\,U_1$, $B_1=\mathrm{Re}\,B\in E'\subset\mathfrak{b}_1$, $B_2=\mathrm{Im}\,B\in E'\subset\mathfrak{b}_1$. Extending Φ complex linearly we obtain

$$0\neq\langle\alpha(X,U_1),B\rangle=\langle\rho(L)\Phi(U_1)-\Phi([L,U_1]),B\rangle=\langle(\rho(L)-\lambda^1(L))\Phi(U_1),B\rangle=0$$

since $B\in E_{-\overline{\lambda^1}}$, which is a contradiction. □

Finally we will assume that ad_0 is not semi-simple. Then ad_0 has a real weight $\lambda\neq 0$ with generalised weight space, i.e. $(\mathrm{ad}_0(L)-\lambda(L))^2=0$ for all $L\in\mathfrak{l}$ but $\mathrm{ad}_0-\lambda\,\mathrm{Id}\neq 0$. We will prove that $\mathfrak{l}$ is not admissible. Assume $\mathfrak{l}$ is admissible and $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp$ for a suitable orthogonal $\mathfrak{l}$-module $\mathfrak{a}$. Let $U\in R$ be a weight vector for λ. Then $R_2:=R_2(\mathfrak{l})=\mathbb{R}\cdot U$. As above one proves that any cocycle $\alpha\in Z^2(\mathfrak{l},\mathfrak{a})$ satisfies $\alpha(R,R)\subset E_{2\lambda}$. Furthermore, there exists a cocycle $\tilde\alpha\in Z^2(\mathfrak{l},\mathfrak{a})$ such that $[\alpha]=[\tilde\alpha]\in H^2(\mathfrak{l},\mathfrak{a})$ and $\tilde\alpha(R_2,\mathfrak{l})\subset E_{2\lambda}+E_\lambda$. Therefore we may assume $\alpha(R_2,\mathfrak{l})\subset E_{2\lambda}+E_\lambda$ and hence $E_{2\lambda}+E_\lambda\subset\mathfrak{b}_2$. Since $\mathfrak{b}_2$ is non-degenerate and $\lambda\neq 0$ we obtain $E_{-2\lambda}+E_{-\lambda}\subset\mathfrak{b}_2$. Because of $E_{-2\lambda}\subset\mathfrak{b}_2$ now Condition (B_2) gives $\alpha|_{R\times R}=0$. By Lemma 6.1 we may

assume $\alpha(R_2,\mathfrak{l}) \subset E_\lambda$. Now $E_{-\lambda} \subset \mathfrak{b}_2$ implies $\alpha(R_2,\mathfrak{l}) = 0$, which is a contradiction to (A_1).

Proof of Proposition 6.2. By the above we know that ad_0 is semi-simple and the weights λ^1 and λ^2 of ad_0 satisfy $\lambda^1+\lambda^2 = 0$ or (possibly after change of the numbering) $\lambda^1+2\lambda^2 = 0$. In particular, $\lambda^1, \lambda^2 \neq 0$, which implies that ad_0 does not have invariants. Therefore we have $H^2(\mathfrak{l}/R, R) = 0$. Hence there exists an abelian subalgebra $\mathfrak{l}_1 \subset \mathfrak{l}$ such that $\mathfrak{l} = \mathfrak{l}_1 \,{}_{\mathrm{ad}_0|_{\mathfrak{l}_1}}\!\ltimes R$. Since λ^1 and λ^2 are linearly dependent the codimension of $\ker(\mathrm{ad}_0|_{\mathfrak{l}_1})$ in $\mathfrak{l}_1$ is one. Hence we may choose an element $X_1 \in \mathfrak{l}_1 \setminus \ker(\mathrm{ad}_0|_{\mathfrak{l}_1})$ such that $\mathfrak{l} = (\mathbb{R}\cdot X_1 \,{}_{\mathrm{ad}_0}\!\ltimes R) \oplus \mathbb{R}^k$ and

$$\mathrm{ad}_0(X_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ or } \mathrm{ad}_0(X_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ or } \mathrm{ad}_0(X_1) = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

with respect to a suitable basis of R. □

For the sake of completeness we will also give the classification result in the nilpotent case. However, we will omit the proof since on the one hand we will not use the result in this paper and on the other hand our proof is tricky and not very enlightening.

Proposition 6.3 *If $\mathfrak{l}$ is an admissible nilpotent Lie algebra with $\dim R(\mathfrak{l}) = 2$, then $\mathfrak{l}$ is isomorphic to one of the following Lie algebras*

$$\begin{aligned}
&\{[X_1,Z]=Y,\ [X_1,X_2]=Z\}\oplus\mathbb{R}^k,\\
&\{[X_1,X_2]=Y,\ [X_1,X_3]=Z\}\oplus\mathbb{R}^k,\\
&\{[X_1,X_2]=Y,\ [X_3,X_4]=Z\}\oplus\mathbb{R}^k \ =\ \mathfrak{h}(1)\oplus\mathfrak{h}(1)\oplus\mathbb{R}^k,\\
&\{[X_1,X_2]=Y,\ [X_1,X_3]=Z,\ [X_3,X_4]=Y\}\oplus\mathbb{R}^k,\\
&\{[X_1,X_2]=Y,\ [X_1,X_3]=Z,\ [X_2,X_4]=Z,\ [X_3,X_4]=\pm Y\}\oplus\mathbb{R}^k.
\end{aligned}$$

7 Metric Lie algebras of index 3

The aim of this section is the classification of all indecomposable metric Lie algebras of index 3.

7.1 Preliminaries

Proposition 7.1 *If $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ is a non-simple indecomposable metric Lie algebra of index 3, then $\mathfrak{g}/\mathfrak{j}(\mathfrak{g})$ is isomorphic to one of the Lie algebras $\mathfrak{n}(2)$, $\mathfrak{r}_{3,-1}$, $\mathfrak{h}(1)$, $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}(2)$ or $\mathbb{R}^k$, $k = 1,2,3$.*

Proof. Let $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ be a non-simple indecomposable metric Lie algebra of index 3. By Proposition 4.1 and Proposition 3.4 $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ has the structure of a balanced quadratic extension $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$, where $\mathfrak{l} = \mathfrak{g}/\mathfrak{j}(\mathfrak{g})$. In particular, $\mathfrak{l} = \mathfrak{g}/\mathfrak{j}(\mathfrak{g})$ is admissible. First we will determine all admissible $\mathfrak{l}$ which can appear in such a quadratic extension. Obviously $\dim\mathfrak{l} \le 3$ because $\langle\cdot,\cdot\rangle$ has index 3. In particular, $\dim R(\mathfrak{l}) \in \{0,1,2\}$. If

$\dim R(\mathfrak{l}) = 0$, then $\mathfrak{l}$ is abelian or simple. If $\dim R(\mathfrak{l}) = 1$, then $\mathfrak{l} \cong \mathfrak{h}(1)$ by Proposition 6.1. Finally, suppose $\dim R(\mathfrak{l}) = 2$. Then $\dim \mathfrak{l} = 3$. In particular, the codimension of $R(\mathfrak{l}) = \mathfrak{l}'$ in $\mathfrak{l}$ is one and therefore $[\mathfrak{l}, \mathfrak{l}'] = [\mathfrak{l}, \mathfrak{l}]$, hence $\mathfrak{l}$ is not nilpotent. Now Proposition 6.2 implies that $\mathfrak{l}$ is one of the 3-dimensional Lie algebras $\mathfrak{n}(2)$, $\mathfrak{r}_{3,-1}$, $\mathfrak{r}_{3,-2}$.

On the other hand we must take into consideration that $\mathfrak{a}$ must be Euclidean if $\mathfrak{l}$ is three-dimensional. This excludes $\mathfrak{l} = \mathfrak{r}_{3,-2}$. Indeed, the weights λ^1 and λ^2 of the representation ad_0 of $\mathfrak{r}_{3,-2}$ are given by $\lambda^1(X) = -2$ and $\lambda^2(X) = 1$. In particular they are real and satisfy λ^1, $\lambda^2 \neq 0$ and $\lambda^1 + \lambda^2 \neq 0$. Hence, if $\mathfrak{a}$ is Euclidean, then $E_{\lambda^1+\lambda^2} + E_{\lambda^1} + E_{\lambda^2} = 0$. If we now assume that $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}, \rho)$ is balanced, then this together with Lemma 6.2 implies $[\alpha] = 0$ in $H^2(\mathfrak{l}, \mathfrak{a})$. Hence $R(\mathfrak{l}) = 0$ by Lemma 6.3, which is a contradiction. □

It remains to classify all indecomposable metric Lie algebras $(\mathfrak{g}, \langle \cdot\,, \cdot \rangle)$ of index 3 for which $\mathfrak{g}/\mathfrak{j}(\mathfrak{g})$ is isomorphic to one of the Lie algebras $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}(2)$, $\mathfrak{n}(2)$, $\mathfrak{r}_{3,-1}$, $\mathfrak{h}(1)$ or $\mathbb{R}^k$ for $k \leq 3$. Before we will start we prove the following fact on 3-dimensional Lie algebras.

Lemma 7.1 *Let $\mathfrak{l}$ be a 3-dimensional Lie algebra such that $R := R(\mathfrak{l})$ is 2-dimensional and ad_0 is semi-simple. We denote the weights of the complexified representation ad_0 by λ^1 and λ^2. Let $\mathfrak{a}$ be a semi-simple orthogonal $\mathfrak{l}$-module and V_λ be defined as above. Choose an element $X \in \mathfrak{l} \setminus R$. Then*

$$\{\alpha \in C^2(\mathfrak{l}, \mathfrak{a}) \mid \alpha(R,R) \subset E_{\lambda^1+\lambda^2},\ \alpha(X, V_{\lambda^i}) \subset E_{\lambda^i},\ i = 1,2\} \longrightarrow H^2(\mathfrak{l}, \mathfrak{a})$$
$$\alpha \longmapsto [\alpha]$$

is well-defined and an isomorphism.

Proof. The map is well defined since $\alpha(R,R) \subset E_{\lambda^1+\lambda^2}$ implies $\alpha \in Z^2(\mathfrak{l}, \mathfrak{a})$. Let us prove that the map is surjective. For a given cohomology class $a = [\alpha] \in H^2(\mathfrak{l}, \mathfrak{a})$ we define cocycles α_1, $\alpha_2 \in Z^2(\mathfrak{l}, \mathfrak{a})$ by

$$\alpha = \alpha_1 + \alpha_2, \quad \alpha_1(X, \cdot) = 0, \quad \alpha_2|_{R\times R} = 0\,.$$

Then $\alpha_1(R,R) \subset E_{\lambda^1+\lambda^2}$ and by Lemma 6.1 there is a cocycle $\tilde\alpha_2 \in Z^2(\mathfrak{l}, \mathfrak{a})$ such that $[\alpha_2] = [\tilde\alpha_2] \in H^2(\mathfrak{l}, \mathfrak{a})$ and $\tilde\alpha_2|_{R\times R} = 0$, $\tilde\alpha_2(\mathfrak{l}, V_{\lambda^i}) \subset E_{\lambda^i}$ for $i = 1,2$. Then $\alpha_1 + \tilde\alpha_2$ is a preimage of a.

It remains to show that the map is injective. Assume that $\alpha = d\tau$ satisfies $d\tau(X, U_i) \in E_{\lambda^i}$ for $U_i \in V_{\lambda^i}$, $i = 1,2$. Since $d\tau(X, U_i) = (\rho(X) - \lambda^i)(\tau(U_i))$ we have on the other hand $p_{\lambda^i}(d\tau(X, U_i)) = 0$. Hence, $d\tau(X, U_i) = 0$ and therefore $\alpha = d\tau = 0$. □

Remark 7.1 The assertion of Lemma 7.1 can also be verified using the Hochschild–Serre spectral sequence associated with the ideal $R \subset \mathfrak{l}$. For dimensional reasons this spectral sequence leads to the exact sequence

$$0 \longrightarrow H^1(\mathfrak{l}/R, H^1(R, \mathfrak{a})) \longrightarrow H^2(\mathfrak{l}, \mathfrak{a}) \longrightarrow H^0(\mathfrak{l}/R, H^2(R, \mathfrak{a})) \longrightarrow 0.$$

Since $H^1(R,\mathfrak{a}) \cong C^1(R,\mathfrak{a}^R) = C^1(R,\mathfrak{a})$ we have

$$\begin{aligned} H^1(\mathfrak{l}/R, H^1(R,\mathfrak{a})) &\cong C^1(\mathbb{R}\cdot X, C^1(R,\mathfrak{a})^{\mathfrak{l}/R}) \\ &\cong \{\alpha \in C^2(\mathfrak{l},\mathfrak{a}) \mid \alpha(X, V_{\lambda^i}) \subset E_{\lambda^i}, i = 1,2\}. \end{aligned}$$

On the other hand

$$\begin{aligned} H^0(\mathfrak{l}/R, H^2(R,\mathfrak{a})) &\cong H^2(R,\mathfrak{a})^{\mathfrak{l}/R} \cong C^2(R,\mathfrak{a})^{\mathfrak{l}/R} \\ &\cong \{\alpha \in C^2(R,\mathfrak{a}) \mid \alpha(R,R) \subset E_{\lambda^1+\lambda^2}\} \end{aligned}$$

and we obtain the assertion of Lemma 7.1.

Lemma 7.2 *Let $\mathfrak{l}$ be a 3-dimensional unimodular Lie algebra. For an orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ the map*

$$\begin{aligned} \iota_Q :\ \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}) &\longrightarrow (\,H^2(\mathfrak{l},\mathfrak{a}) \setminus \{0\}\,) \ \cup\ C^3(\mathfrak{l}) \\ [\alpha,\gamma] &\longmapsto \begin{cases} [\alpha] \in H^2(\mathfrak{l},\mathfrak{a}) & \textit{if } [\alpha] \neq 0 \\ \gamma \in C^3(\mathfrak{l}) & \textit{if } [\alpha] = 0 \end{cases} \end{aligned}$$

is a bijection.

Proof. To prove this we will use Proposition 2.3. Let us determine the vector space $H^3(\mathfrak{l})/a \cup H^1(\mathfrak{l},\mathfrak{a})$ for $a = [\alpha] \in H^2_\cup(\mathfrak{l},\mathfrak{a}) = H^2(\mathfrak{l},\mathfrak{a})$. Since $\mathfrak{l}$ is unimodular $d\sigma(X,Y,Z) = -\operatorname{tr}(\mathrm{ad}_0(X)) \cdot \sigma(Y,Z) = 0$ holds for all $\sigma \in C^2(\mathfrak{l},\mathfrak{a})$. Hence $H^3(\mathfrak{l}) \cong C^3(\mathfrak{l})$ is one-dimensional. By (5) we know that $\cup : H^2(\mathfrak{l},\mathfrak{a}) \otimes H^1(\mathfrak{l},\mathfrak{a}) \to H^3(\mathfrak{l})$ is a non-degenerate pairing. If now $a \neq 0$, then this implies $H^3(\mathfrak{l})/a \cup H^1(\mathfrak{l},\mathfrak{a}) = 0$. Hence for $a \neq 0$ the set $p^{-1}(a)$ only consists of one element, namely $[\alpha, 0] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$. Now suppose $a = 0$. Then we have $H^3(\mathfrak{l})/0 \cup H^1(\mathfrak{l},\mathfrak{a}) = H^3(\mathfrak{l}) \cong C^3(\mathfrak{l})$. Hence, $p^{-1}(0) \ni [0,\gamma] \mapsto \gamma \in C^3(\mathfrak{l})$ is a bijection. □

Let $\mathbb{R}^{p,q}$ be the standard pseudo-Euclidean space of dimension $n = p + q$. As usual, $\mathbb{R}^n := \mathbb{R}^{0,n}$. We identify $\mathbb{R}^{p_1,q_1} \oplus \mathbb{R}^{p_2,q_2}$ with $\mathbb{R}^{p_1+p_2,q_1+q_2}$.

Definition 7.1 *Let $\mathfrak{l}_0$ be an abelian Lie algebra and $\lambda \in \mathfrak{l}_0^*$. We define orthogonal representations ρ^+_λ of $\mathfrak{l}_0$ on $\mathbb{R}^2$, ρ^-_λ of $\mathfrak{l}_0$ on $\mathbb{R}^{2,0}$ and ρ'_λ of $\mathfrak{l}_0$ on $\mathbb{R}^{1,1}$ by*

$$\rho^\pm_\lambda(L) = \begin{pmatrix} 0 & -\lambda(L) \\ \lambda(L) & 0 \end{pmatrix}, \quad \rho'_\lambda(L) = \begin{pmatrix} 0 & \lambda(L) \\ \lambda(L) & 0 \end{pmatrix}$$

w. r. t. an orthonormal basis of $\mathbb{R}^2$, $\mathbb{R}^{2,0}$, and $\mathbb{R}^{1,1}$, respectively.
Moreover, for μ, $\nu \in \mathfrak{l}_0^*$ we define an orthogonal representation $\rho''_{\mu,\nu}$ of $\mathfrak{l}_0$ on $\mathbb{R}^{2,2}$ by

$$\rho''_{\mu,\nu}(L) = \begin{pmatrix} 0 & -\nu(L) & \mu(L) & 0 \\ \nu(L) & 0 & 0 & \mu(L) \\ \mu(L) & 0 & 0 & -\nu(L) \\ 0 & \mu(L) & \nu(L) & 0 \end{pmatrix}$$

w. r. t. an orthonormal basis of $\mathbb{R}^{2,2}$.

For $\lambda = (\lambda^1,\dots,\lambda^m)$, $\mu = (\mu^1,\dots,\mu^m)$, $\nu = (\nu^1,\dots,\nu^m) \in (\mathfrak{l}_0^*)^m$ we define semi-simple orthogonal representations ρ_λ^+ of $\mathfrak{l}_0$ on $\mathbb{R}^{2m}$, ρ_λ^- of $\mathfrak{l}_0$ on $\mathbb{R}^{2m,0}$, ρ'_λ of $\mathfrak{l}_0$ on $\mathbb{R}^{m,m}$ and $\rho''_{\mu,\nu}$ of $\mathfrak{l}_0$ on $\mathbb{R}^{2m,2m}$ by

$$\rho_\lambda^\pm = \bigoplus_{i=1}^m \rho_{\lambda^i}^\pm\,, \quad \rho'_\lambda = \bigoplus_{i=1}^m \rho'_{\lambda^i}\,, \quad \rho''_{\mu,\nu} = \bigoplus_{i=1}^m \rho''_{\mu^i,\nu^i}.$$

Now let $\mathfrak{l}$ be a solvable Lie algebra, $\mathfrak{l}_0 = \mathfrak{l}/R(\mathfrak{l})$ and let ρ_λ^+, ρ_λ^-, ρ'_λ and $\rho''_{\mu,\nu}$ be the above defined representation of $\mathfrak{l}_0$. Composing the projection $\mathfrak{l} \to \mathfrak{l}_0$ with these representations we obtain representations of $\mathfrak{l}$, which we denote by the same symbol.

Finally we denote by ρ_0 the trivial representation of $\mathfrak{l}$ on $\mathfrak{a} = \mathbb{R}^{p,q}$.

The symmetric group $\mathfrak{S}_m$ acts on $(\mathfrak{l}_0^*)^m$ by permuting coordinates and on $(\mathfrak{l}_0^*)^m \oplus (\mathfrak{l}_0^*)^m$ by permuting pairs of coordinates. The group $(\mathbb{Z}_2)^m$ acts on $(\mathfrak{l}_0^*)^m$ by changing the signs of the coordinates. We define the orbit spaces $\Lambda_m := (\mathfrak{l}_0^* \setminus 0)^m/\mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m$ and $\Lambda''_m := \Big(((\mathfrak{l}_0^* \setminus 0)^m/(\mathbb{Z}_2)^m) \oplus ((\mathfrak{l}_0^* \setminus 0)^m/(\mathbb{Z}_2)^m)\Big)/\mathfrak{S}_m$. Finally we define an action of $\mathrm{Aut}(\mathfrak{l})$ on Λ_m and Λ''_m by $S^*[\lambda] := [S^*\lambda]$ and $S^*[\mu,\nu] := [S^*\mu, S^*\nu]$.

Proposition 7.2 *For a solvable Lie algebra $\mathfrak{l}$ we consider the map*

$$\begin{array}{ccc} \displaystyle\bigcup_{\substack{(m_1,\dots,m_4,p_0,q_0)\\ 2m_1+m_3+2m_4+p_0=p\\ 2m_2+m_3+2m_4+q_0=q}} \Lambda_{m_1}\times\Lambda_{m_2}\times\Lambda_{m_3}\times\Lambda''_{m_4} & \longrightarrow & \mathrm{Hom}(\mathfrak{l},\mathfrak{so}(p,q))_{\mathrm{ss}}/O(p,q) \\ ([\lambda_1],[\lambda_2],[\lambda_3],[\mu,\nu]) & \longmapsto & [\rho_{\lambda_1}^+ \oplus \rho_{\lambda_2}^- \oplus \rho'_{\lambda_3} \oplus \rho''_{\mu,\nu} \oplus \rho_0]\,, \end{array}$$

where ρ_0 is the trivial representation of $\mathfrak{l}$ on $\mathbb{R}^{p_0,q_0}$. This map is a bijection. It is equivariant with respect to the action of $\mathrm{Aut}(\mathfrak{l})$*.*

We call an orthogonal basis $A_1,\dots,A_n$ of $\mathbb{R}^{p,q}$, $p+q=n$, orthonormal if $\langle A_k,A_k\rangle = -1$ for $k=1,\dots,p$ and $\langle A_k,A_k\rangle = 1$ for $k=p+1,\dots,n$. For later use we fix an orthonormal basis $A_1^+,\dots,A_{2m}^+$ of the $\mathfrak{l}$-module $(\rho_\lambda^+,\mathbb{R}^{2m})$ and an orthonormal basis $A_1^0,\dots,A_{p+q}^0$ of the trivial $\mathfrak{l}$-module $(\rho_0,\mathbb{R}^{p,q})$.

7.2 The case $\mathfrak{l} = \mathfrak{n}(2)$

First we consider $\mathfrak{l} = \mathfrak{n}(2)$. We will write elements of $\mathrm{Aut}(\mathfrak{l})$ as matrices w. r. t. the basis X, Y, Z of $\mathfrak{l}$.

$$\mathrm{Aut}(\mathfrak{l}) = \left\{ S(u,v,w,a,b) := \begin{pmatrix} u & 0 & 0 \\ v & a & -b \\ w & ub & ua \end{pmatrix} \;\Big|\; u=\pm1,\ a,b,v,w\in\mathbb{R},\ a^2+b^2\neq 0 \right\}$$

Let λ and $\bar\lambda = -\lambda$ be the weights of the representation ad_0 of $\mathfrak{l}_\mathbb{C}/R_\mathbb{C}$ on $R_\mathbb{C}$. Then $\lambda(X) = i$. We use again the notation $E = (E_\lambda \oplus E_{\bar\lambda}) \cap \mathfrak{a} \subset \mathfrak{a}_\mathbb{C}$. Next we will determine

$H^2(\mathfrak{l},\mathfrak{a})$, $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$, and $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp$ for a Euclidean semi-simple orthogonal $\mathfrak{l}$-module $(\rho,\mathfrak{a})$. By Lemma 7.1 the map

$$\{\alpha \in C^2(\mathfrak{l},\mathfrak{a}) \mid \alpha(Y,Z)\in \mathfrak{a}^{\mathfrak{l}}, \alpha(X,Y)\in E, \alpha(X,Z)=\rho(X)\alpha(X,Y)\} \longrightarrow H^2(\mathfrak{l},\mathfrak{a})$$
$$\alpha \longmapsto [\alpha]$$

is an isomorphism. We denote the inverse of this isomorphism by ι.

Lemma 7.3 *If $\mathfrak{a}$ is a Euclidean orthogonal $\mathfrak{l}$-module, then we have*

$$\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp = \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}) \setminus \{[0,0]\} = \iota_Q^{-1}(\,(H^2(\mathfrak{l},\mathfrak{a})\setminus 0)\cup(C^3(\mathfrak{l})\setminus 0)\,).$$

Proof. We have to check under which conditions a cohomology class $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ is admissible. First we note that (A_0) and (A_k), $k>1$ are satisfied for all $[\alpha,\gamma]$ since $\mathfrak{z}(\mathfrak{l})=0$ and $R_k(\mathfrak{l})=0$ for $k>1$. Conditions (B_0) and (B_k), $k\ge 1$ are also satisfied since $\mathfrak{a}$ is Euclidean. It remains to check (A_1) for $\mathfrak{k}=R$ since R is the only non-vanishing ideal in R. We may assume $\alpha=\iota([\alpha])$. In particular, $\alpha(X,Y-iZ)\in E_\lambda$. On the other hand $(A_1)(i)$ is satisfied if and only if $\alpha(Y,Z)=0$ and $\alpha(X,Y-iZ)=(\rho(X)-i)\Phi_1(X-iY)$ for a homomorphism $\Phi_1\in\mathrm{Hom}(R,\mathfrak{a})$, thus if and only if $\alpha=0$. Hence all $[\alpha,\gamma]$ with $[\alpha]\neq 0$ are admissible. Now we assume $\alpha=0$. Obviously, $[0,0]$ is not admissible. Suppose that $\gamma\neq 0$. Assume that $[0,\gamma]$ satisfies the assumption (ii) of (A_1). Then there is a homomorphism $\Phi_2\in\mathrm{Hom}(\mathfrak{k},R^*)$ such that

$$\gamma(X,Y,Z)=\langle\Phi_2(Y),[X,Z]\rangle+\langle\Phi_2([X,Y]),Z\rangle=-\langle\Phi_2(Y),Y\rangle+\langle\Phi_2(Z),Z\rangle$$

and

$$\gamma(X,Z,Y)=\langle\Phi_2(Z),[X,Y]\rangle+\langle\Phi_2([X,Z]),Y\rangle=\langle\Phi_2(Z),Z\rangle-\langle\Phi_2(Y),Y\rangle.$$

Since on the other hand $\gamma(X,Y,Z)=-\gamma(X,Z,Y)$ these equations imply $\gamma=0$, a contradiction. Thus $[0,\gamma]$ does not satisfy the assumption (ii) of (A_1). Consequently, the implication (A_1) is true in this case. Hence, $[0,\gamma]$ is admissible for all $\gamma\neq 0$. $\square$

Since here $\mathfrak{l}_0=\mathfrak{l}/R(\mathfrak{l})=\mathbb{R}\cdot X$ we identify $\lambda\in(\mathfrak{l}_0^*)^m$ with $\lambda(X)\in\mathbb{R}^m$.

Proposition 7.3 *If $(\mathfrak{g},\langle\cdot\,,\cdot\rangle)$ is an indecomposable metric Lie algebra of index 3 such that $\mathfrak{g}/\mathfrak{j}(\mathfrak{g})\cong\mathfrak{l}:=\mathfrak{n}(2)$, then $\mathfrak{g}$ is isomorphic to exactly one of the following indecomposable metric Lie algebras $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$:*

(Ia) $\mathfrak{a}=\mathbb{R}^{2m}$, $m\ge 0$, $\rho=\rho_\lambda^+$,
where $\lambda=(\lambda^1,\dots,\lambda^m)$, $0<\lambda^1\le\lambda^2\le\dots\le\lambda^m$,
$\alpha=0$,
$\gamma(X,Y,Z)=1$;

(Ib) *as above but* $\gamma(X,Y,Z)=-1$;

(II) $\mathfrak{a}=\mathbb{R}^{2m+1}=\mathbb{R}^{2m}\oplus\mathbb{R}^1$, $m\ge 0$, $\rho=\rho_\lambda^+\oplus\rho_0$,
where $\lambda=(\lambda^1,\dots,\lambda^m)$, $0<\lambda^1\le\lambda^2\le\dots\le\lambda^m$,
$\alpha(Y,Z)=A_1^0$, $\alpha(X,Y)=\alpha(X,Z)=0$,
$\gamma=0$;

(III) $\mathfrak{a} = \mathbb{R}^{2m+3} = \mathbb{R}^{2m+2} \oplus \mathbb{R}^1$, $m \geq 0$, $\rho = \rho^+_{\lambda'} \oplus \rho_0$,
where $\lambda' = (\lambda^1, \dots, \lambda^m, 1)$, $0 < \lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^m$,
$\alpha(Y,Z) = A^0_1$, $\alpha(X,Y) = rA^+_{2m+1}$, $\alpha(X,Z) = rA^+_{2m+2}$, $r > 0$;
$\gamma = 0$;

(IV) $\mathfrak{a} = \mathbb{R}^{2m+2}$, $m \geq 0$, $\rho = \rho^+_{\lambda'}$,
where $\lambda' = (\lambda^1, \dots, \lambda^m, 1)$, $0 < \lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^m$,
$\alpha(Y,Z) = 0$, $\alpha(X,Y) = A^+_{2m+1}$, $\alpha(X,Z) = A^+_{2m+2}$;
$\gamma = 0$.

Proof. We already know that $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ is a quadratic extension of $\mathfrak{l}$ by a suitable orthogonal $\mathfrak{l}$-module $\mathfrak{a}$. Since $\langle\cdot\,,\cdot\rangle$ has index 3, $\mathfrak{a}$ must be Euclidean, i.e. $\mathfrak{a} = \mathbb{R}^n$. In particular we have $\mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(\mathfrak{a}, \langle\cdot\,,\cdot\rangle_{\mathfrak{a}}))_{\mathrm{ss}} = \mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(n))$. If $\rho \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(n))$ and $S \in \mathrm{Aut}(\mathfrak{l})$, then $\rho \circ S = \pm\rho$ holds. On the other hand, if $\rho \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(n))$, then there exists a map $U \in O(n)$ such that $\mathrm{Ad}(U)\rho = -\rho$. This implies

$$\mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(n))/G = \mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(n))/(\mathbb{Z}_2 \times O(n)) = \mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(n))/O(n).$$

By Proposition 7.2 the map

$$\begin{array}{ccc} \bigcup_{2m \leq n} \Lambda_m & \longrightarrow & \mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(n))/G \\ [\lambda] & \longmapsto & [\rho^+_\lambda \oplus \rho_0] \end{array}$$

is a bijection. Since we can identify $\lambda \in \Lambda_m$ with $\lambda(X) \in \mathbb{R}^m$, we can also identify

$$\Lambda_m = \{\lambda \in \mathbb{R}^m \mid 0 < \lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^m\}.$$

Hence each G-orbit in $\mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(n))$ has a canonical representative $\rho^+_\lambda \oplus \rho_0$ with $\lambda \in \Lambda_m$. We fix a representation $\rho := \rho^+_\lambda \oplus \rho_0$. Next we will describe the action of G_ρ on $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}_\rho)$ identifying $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}_\rho)$ with $(\, H^2(\mathfrak{l}, \mathfrak{a}_\rho) \setminus \{0\}\,) \cup C^3(\mathfrak{l})$ via ι_Q and $H^2(\mathfrak{l}, \mathfrak{a}_\rho) \setminus \{0\}$ with $\iota(H^2(\mathfrak{l}, \mathfrak{a}_\rho) \setminus \{0\})$. We claim that for $(S, U) \in G_\rho$ with $S = S(u, v, w, a, b)$ and $\alpha \in \iota(\, H^2(\mathfrak{l}, \mathfrak{a}_\rho) \setminus \{0\}\,)$, $\gamma \in C^3(\mathfrak{l})$ the following holds:

$$\begin{aligned}
\iota \circ \iota_Q((S,U)^* \iota_Q^{-1}([\alpha])) &= \tilde{\alpha} \quad \text{with} \\
\tilde{\alpha}(Y,Z) &= u(a^2+b^2)U^{-1}(\alpha(Y,Z)) \in \mathfrak{a}^{\mathfrak{l}} && (49) \\
\tilde{\alpha}(X,Y) &= u(a+b\rho(X))U^{-1}(\alpha(X,Y)) \in E && (50) \\
\tilde{\alpha}(X,Z) &= \rho(X)(\tilde{\alpha}(X,Y)) && (51) \\
\iota_Q((S,U)^* \iota_Q^{-1}(\gamma)) &= (a^2+b^2)\gamma. && (52)
\end{aligned}$$

Let us verify this. The condition $(S, U) \in G_\rho$ says that $\mathrm{Ad}(U^{-1})(\rho(S(X))) = \rho(X)$ holds, which here is equivalent to $U^{-1}\rho(uX)U = \rho(X)$. We note that on the one hand

$$S(u, v, w, a, b) = S(1, uv, uw, 1, 0) \cdot S(u, 0, 0, a, b)$$

and that on the other hand by Proposition 2.2 the subgroup

$$G'_\rho := \{(S(1, v, w, 1, 0), \mathrm{Id}) \mid v, w \in \mathbb{R}\} \subset \{(e^{\mathrm{ad}(L)}, e^{\rho(L)}) \mid L \in \mathfrak{l}\} \subset G_\rho$$

acts trivially on $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)$. Therefore it suffices to prove

$$\iota\circ\iota_Q((S_0,U)^*\iota_Q^{-1}([\alpha]))=\tilde\alpha,\quad \iota_Q((S_0,U)^*\iota_Q^{-1}(\gamma))=(a^2+b^2)\gamma$$

for $S_0=S(u,0,0,a,b)$. We have

$$\iota\circ\iota_Q((S_0,U)^*\iota_Q^{-1}([\alpha]))=\iota_Q(U_*^{-1}S_0^*[\alpha,0])=\iota_Q([U_*^{-1}S_0^*\alpha,0])=\iota(U_*^{-1}S_0^*\alpha)$$

and we calculate

$$\begin{aligned}(U_*^{-1}S_0^*\alpha)(Y,Z)&=U^{-1}(\alpha(S_0Y,S_0Z))\\&=u(a^2+b^2)U^{-1}(\alpha(Y,Z))\\(U_*^{-1}S_0^*\alpha)(X,Y)&=U^{-1}(\alpha(S_0X,S_0Y))\\&=U^{-1}(au\,\alpha(X,Y)+u^2b\,\alpha(X,Z))\\&=U^{-1}(u(a+b\rho(uX))\alpha(X,Y))\\&=u(a+b\rho(X))U^{-1}(\alpha(X,Y))\\(U_*^{-1}S_0^*\alpha)(X,Z)&=u(-b+a\rho(X))U^{-1}(\alpha(X,Y)).\end{aligned}$$

Then $\tilde\alpha:=U_*^{-1}S_0^*\alpha$ satisfies (49), (50), and (51). Hence, $\tilde\alpha=\iota(U_*^{-1}S_0^*\alpha)$. Finally,

$$\iota_Q((S_0,U)^*\iota_Q^{-1}(\gamma))=S_0^*\gamma=\det S_0\cdot\gamma=(a^2+b^2)\gamma,$$

which proves the claim.

Using this description of the G_ρ-action we can distinguish between the following types of G_ρ-orbits in $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_0$, which we characterise by properties of their elements $[\alpha,\gamma]$, where we may assume that $\alpha=\iota([\alpha])$ and, moreover, that $\gamma=0$ if $\alpha\neq0$:

Type (Ia): $\alpha=0,\ \gamma(X,Y,Z)>0$,
Type (Ib): $\alpha=0,\ \gamma(X,Y,Z)<0$,
Type (II): $\alpha(Y,Z)\neq0,\ \alpha(X,Y)=\alpha(X,Z)=0,\ \gamma=0$,
Type (III): $\alpha(Y,Z)\neq0,\ \alpha(X,\cdot)\neq0$ on R, $\gamma=0$,
Type (IV) : $\alpha(Y,Z)=0,\ \alpha(X,\cdot)\neq0$ on R, $\gamma=0$.

Next we will classify the G_ρ-orbits of each type. The result will give the Lie algebras in the corresponding item of the proposition.

Since for $\mathfrak{l}=\mathfrak{n}(2)$ each decomposition $\mathfrak{l}=\mathfrak{l}_1\oplus\mathfrak{l}_2$ of Lie algebras is trivial Definition 5.1 says that a cohomology class $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_\sharp$ is decomposable if and only if $\mathfrak{a}_\rho^{\mathfrak{l}}\cap(\alpha(\mathfrak{l},\mathfrak{l}))^\perp=0$. Therefore $[\alpha,\gamma]\in\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_\sharp$ is indecomposable if and only if $\mathfrak{a}_\rho^{\mathfrak{l}}=\mathbb{R}\cdot\alpha(Y,Z)$. In particular, we may assume $n=2m$ or $n=2m+1$ since otherwise $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_0$ is empty.

We start with orbits of type (Ia) and type (Ib). Here we may assume $n=2m$. If $S=S(1,0,0,a,b)\in\mathrm{Aut}(\mathfrak{l})$ then $(S,\mathrm{Id})\in G_\rho$. This together with (52) implies that two elements $[0,\gamma_1]$ and $[0,\gamma_2]$ of $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_0$ are in the same G_ρ-orbit of type (Ia) (or of type (Ib)) if and only if $\gamma_1(X,Y,Z)$ and $\gamma_2(X,Y,Z)$ have the same sign. This yields a classification of orbits of type (Ia) and type (Ib).

Now we consider orbits of type (II) or (III). Here we may assume $n = 2m+1$. Besides $(S, \mathrm{Id}) \in G_\rho$ for $S = S(1,0,0,a,b) \in \mathrm{Aut}(\mathfrak{l})$ we also have $(S, -\mathrm{Id}) \in G_\rho$. Now (49) implies that each orbit of type (II) or (III) contains an element $[\alpha, 0]$ with $\alpha = \iota([\alpha])$ and $\alpha(Y,Z) = A_1^0$. For G_ρ-orbits of type (II) this yields the claimed classification. Now consider elements $[\alpha_1, 0]$, $[\alpha_2, 0]$ which belong to G_ρ-orbits of type (III) and satisfy $\alpha_i = \iota([\alpha_i])$ and $\alpha_i(Y,Z) = A_1^0$, $i = 1,2$. Assume $[\alpha_1, 0]$ and $[\alpha_2, 0]$ are in the same G_ρ-orbit, i.e. there is an element $(S,U) \in G_\rho$, $S = S(u,v,w,a,b)$ such that $\iota_Q((S,U)^*\iota_Q^{-1}(\alpha_1)) = \alpha_2$. Then $a^2 + b^2 = 1$ by (49). Now (50) implies

$$|\alpha_1(X,Y)| = |\alpha_2(X,Y)| \tag{53}$$

since ρ is orthogonal and $\rho^2 = -\mathrm{Id}$ on E. Hence, (53) is a necessary condition for $[\alpha_1, 0]$ and $[\alpha_2, 0]$ being in the same G_ρ-orbit. We will show that it is also sufficient. Assume (53) is satisfied. Since, furthermore, $\alpha_i(X,Z) = \rho(X)\alpha_i(X,Y)$ for $i = 1,2$ we can define an orthogonal map U on E which commutes with $\rho(X)$ such that $U(\alpha_2(X,Y)) = \alpha_1(X,Y)$ and $U(\alpha_2(X,Z)) = \alpha_1(X,Z)$. We extend U to a map $U_0 \in O(n)$ such that $U_0|_{E^\perp} = \mathrm{Id}$. Then $(\mathrm{Id}, U_0) \in G_\rho$ and $\iota_Q((\mathrm{Id}, U_0)^*\iota_Q^{-1}(\alpha_1)) = \alpha_2$, hence $[\alpha_1, 0]$ and $[\alpha_2, 0]$ are in the same G_ρ-orbit.

Now consider elements $[\alpha_1, 0]$, $[\alpha_2, 0]$ which belong to G_ρ-orbits of type (IV). As usual we may assume $\alpha_i = \iota([\alpha_i])$ for $i = 1,2$. We set $r_0 := |\alpha_2(X,Y)|/|\alpha_1(X,Y)| = |\alpha_2(X,Z)|/|\alpha_1(X,Z)|$ and define as above a map $U_0 \in O(n)$ commuting with $\rho(X)$ such that $U_0(\alpha_2(X,Y)) = r_0\alpha_1(X,Y)$ and $U_0(\alpha_2(X,Z)) = r_0\alpha_1(X,Z)$. If we choose $S = S(1,0,0,\sqrt{r_0},0)$, then $(S, U_0) \in G_\rho$ and $\iota_Q((S,U_0)^*\iota_Q^{-1}(\alpha_1)) = \alpha_2$. Hence $[\alpha_1, 0]$ and $[\alpha_2, 0]$ are in the same G_ρ-orbit. $\square$

7.3 The case $\mathfrak{l} = \mathfrak{r}_{3,-1}$

Now we suppose $\mathfrak{l} = \mathfrak{r}_{3,-1}$. Let again λ^1 and λ^2 be the weights of the representation ad_0 of $\mathfrak{l}_\mathbb{C}/R_\mathbb{C}$ on $R_\mathbb{C}$. They are given by $\lambda^1(X) = 1$ and $\lambda^2(X) = -1$. In particular both weights are real. Since $\mathfrak{a}$ is Euclidean this implies $E_{\lambda^1} = E_{\lambda^2} = 0$. Moreover, we have $E_{\lambda^1+\lambda^2} = \mathfrak{a}^\mathfrak{l}$. Suppose $\rho \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(\mathfrak{a}, \langle\cdot\,,\cdot\rangle_\mathfrak{a}))_{\mathrm{ss}}$. By Lemma 7.1 the map

$$\begin{aligned} \{\alpha \in C^2(\mathfrak{l},\mathfrak{a}) \mid \alpha(Y,Z) \in \mathfrak{a}^\mathfrak{l}, \alpha(X,\cdot) = 0\} &\longrightarrow H^2(\mathfrak{l},\mathfrak{a}) \\ \alpha &\longmapsto [\alpha] \end{aligned} \tag{54}$$

is an isomorphism. Let ι be the inverse of this isomorphism. Obviously

$$\begin{aligned} \iota_1 : \quad \{\alpha \in C^2(\mathfrak{l},\mathfrak{a}) \mid \alpha(Y,Z) \in \mathfrak{a}^\mathfrak{l}, \alpha(X,\cdot) = 0\} &\longrightarrow \mathfrak{a}^\mathfrak{l} \\ \alpha &\longmapsto \alpha(Y,Z)\,. \end{aligned}$$

is an isomorphism. Since $d\tau(Y,Z) = 0$ for all $\tau \in C^1(\mathfrak{l},\mathfrak{a})$ the map

$$\begin{array}{rccc} \iota_0 : & H^2(\mathfrak{l},\mathfrak{a}) & \longrightarrow & \mathfrak{a}^\mathfrak{l} \\ & [\alpha] & \longmapsto & \alpha(Y,Z) \end{array}$$

is well-defined. We have $\iota_1^{-1} \circ \iota_0 \circ \iota^{-1} = \mathrm{Id}$, thus $\iota_0 = \iota_1 \circ \iota$. In particular, ι_0 is an isomorphism.

Lemma 7.4 *If $\mathfrak{a}$ is a Euclidean orthogonal $\mathfrak{l}$-module, then we have*

$$\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp = \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}) \setminus \{[0,0]\} = \iota_Q^{-1}(\,(H^2(\mathfrak{l},\mathfrak{a})\setminus 0) \cup (C^3(\mathfrak{l})\setminus 0)\,).$$

Proof. We have to check which cohomology classes $[\alpha,\gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ are admissible. We assume $\alpha = \iota([\alpha])$. As in the case of $\mathfrak{l} = \mathfrak{n}(2)$ all conditions but (A_1) are trivially satisfied. Note that here $(A_1)(i)$ is equivalent to $\alpha(Y,Z) = 0$ and thus to $\alpha = 0$. However in case $\alpha = 0$ Condition $(A_1)(ii)$ is satisfied if and only if

$$\gamma(X,Y,Z) = \langle \Phi_2(Y), [X,Z]\rangle + \langle \Phi_2([X,Y]), Z\rangle = \langle \Phi_2(Y), -Z\rangle + \langle \Phi_2(Y), Z\rangle = 0.$$

Hence (A_1) holds for all $[\alpha,\gamma] \neq [0,0]$. Obviously $[0,0]$ is not admissible. $\square$

Next we will describe the automorphism group of $\mathfrak{l} = \mathfrak{r}_{3,-1}$. Let $v,w,a,b,c,d \in \mathbb{R}$, such that $ad \neq 0$ and $bc \neq 0$. We define automorphisms $S'(v,w,a,d)$ and $S''(v,w,b,c)$ of $\mathfrak{l}$ by the following matrices with respect to the basis X, Y, Z of $\mathfrak{l}$:

$$S'(v,w,a,d) = \begin{pmatrix} 1 & 0 & 0 \\ v & a & 0 \\ w & 0 & d \end{pmatrix}, \quad S''(v,w,b,c) = \begin{pmatrix} -1 & 0 & 0 \\ v & 0 & b \\ w & c & 0 \end{pmatrix}.$$

Then we have

$$\mathrm{Aut}(\mathfrak{l}) = \{S'(v,w,a,d) \mid v,w,a,d \in \mathbb{R},\ ad \neq 0\} \cup \{S''(v,w,b,c) \mid v,w,b,c \in \mathbb{R},\ bc \neq 0\}.$$

Since $\mathfrak{l}_0 = \mathfrak{l}/R(\mathfrak{l}) = \mathbb{R}\cdot X$ we identify again $\lambda \in (\mathfrak{l}_0^*)^m$ with $\lambda(X) \in \mathbb{R}^m$.

Proposition 7.4 *If $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ is an indecomposable metric Lie algebra of index 3 such that $\mathfrak{g}/\mathfrak{j}(\mathfrak{g})^\perp \cong \mathfrak{l} := \mathfrak{r}_{3,-1}$, then $\mathfrak{g}$ is isomorphic to exactly one of the following indecomposable metric Lie algebras $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a},\rho)$ with*

(I) $\mathfrak{a} = \mathbb{R}^{2m+1} = \mathbb{R}^{2m} \oplus \mathbb{R}^1$, $m \geq 0$, $\rho = \rho_\lambda^+ \oplus \rho_0$,
where $0 < \lambda^1 \leq \lambda^2 \leq \ldots \leq \lambda^m$,
$\alpha(Y,Z) = A_1^0$, $\alpha(X,Y) = \alpha(X,Z) = 0$,
$\gamma = 0$,

(II) $\mathfrak{a} = \mathbb{R}^{2m}$, $m \geq 0$, $\rho = \rho_\lambda^+$,
where $0 < \lambda^1 \leq \lambda^2 \leq \ldots \leq \lambda^m$,
$\alpha = 0$,
$\gamma(X,Y,Z) = 1$.

Proof. Again $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ is a quadratic extension of $\mathfrak{l}$ by $\mathfrak{a} = \mathbb{R}^n$. By the same reasons as in the proof of Proposition 7.3 each element in $\mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(n))/G$ has a canonical representative $\rho_\lambda^+ \oplus \rho_0$, where $\lambda \in \mathbb{R}^m$, $2m \leq n$, $0 < \lambda^1 \leq \lambda^2 \leq \ldots \leq \lambda^m$ and ρ_0 is the trivial representation on $\mathbb{R}^{n-2m}$. We fix such a representation $\rho = \rho_\lambda^+ \oplus \rho_0$. As in the case of $\mathfrak{n}(2)$ a cohomology class $[\alpha,\gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_\sharp$ is indecomposable if and only if $\mathfrak{a}_\rho^{\mathfrak{l}} = \mathbb{R}\cdot\alpha(Y,Z)$. In particular, we may assume $n = 2m$ or $n = 2m+1$ since otherwise $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_0$ is empty.

Let $S := S'(0,0,a,d)$ be as above. Then we have $(S, \mathrm{Id}) \in G_\rho$ and

$$\begin{aligned} \iota \circ \iota_Q((S,\mathrm{Id})^* \iota_Q^{-1}([\alpha])) &= \iota([S^*\alpha]) = \iota_1^{-1} \circ \iota_0([S^*\alpha)]) = \iota_1^{-1}(ad\, \alpha(Y,Z)) \\ \iota_Q((S,\mathrm{Id})^* \iota_Q^{-1}(\gamma)) &= ad \cdot \gamma. \end{aligned}$$

Therefore G_ρ acts transitively on $\iota_Q^{-1}(H^2(\mathfrak{l},\mathfrak{a}_\rho)\setminus 0)$ and on $\iota_Q^{-1}(C^3(\mathfrak{l})\setminus 0)$. It follows that we can represent each element in $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_0/G_\rho$ as in *(I)* or *(II)*. □

7.4 The case $\mathfrak{l} = \mathfrak{h}(1)$

We consider now $\mathfrak{l} = \mathfrak{h}(1)$. Here we have $R = \mathbb{R} \cdot Z$.

Lemma 7.5 *For $\rho \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(\mathfrak{a}, \langle\cdot,\cdot\rangle_\mathfrak{a}))_{\mathrm{ss}}$ the map*

$$\begin{aligned} \{\alpha \in C^2(\mathfrak{l},\mathfrak{a}) \mid \alpha(X,Y) = 0,\ \alpha(Z,\mathfrak{l}) \subset \mathfrak{a}^\mathfrak{l}\} &\longrightarrow H^2(\mathfrak{l},\mathfrak{a}) \\ \alpha &\longmapsto [\alpha] \end{aligned}$$

is an isomorphism.

Proof. The map is well defined and injective. We will prove that it is surjective. Let $a \in H^2(\mathfrak{l},\mathfrak{a})$ be given. By Lemma 6.1 we know that $a = [\alpha]$ with $\alpha(Z,\mathfrak{l}) \subset \mathfrak{a}^\mathfrak{l}$. We write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1(X,Y) = 0$ and $\alpha_2(Z,\cdot) = 0$. Then α_2 belongs to the subcomplex of $C^*(\mathfrak{l},\mathfrak{a})$ which consists of all cochains $\sigma \in C^*(\mathfrak{l},\mathfrak{a})$ which satisfy $\sigma(Z,\cdot) = 0$. This subcomplex is equivalent to $C^*(\mathfrak{l}/R,\mathfrak{a})$. Since $\mathfrak{l}/R$ is abelian we obtain $[\alpha_2] = [\tilde\alpha_2]$ for a cocycle $\tilde\alpha_2 \in Z^2(\mathfrak{l},\mathfrak{a})$ satisfying $\tilde\alpha_2(X,Y) =: A \in \mathfrak{a}^\mathfrak{l}$ and $\tilde\alpha_2(Z,\cdot) = 0$. Now we define a cocycle $\tau \in Z^1(\mathfrak{l},\mathfrak{a})$ by $\tau(X) = \tau(Y) = 0$, $\tau(Z) = A$. Then $\tilde\alpha_2 + d\tau = 0$. Hence, $[\alpha] = [\alpha_1]$ and we obtain $[\alpha_1] = a$. □

Again the lemma can be considered as a consequence of the Hochschild-Serre spectral sequence associated with $R \subset \mathfrak{l}$ (compare Remark 7.1).

Let ι denote the inverse of the isomorphism defined in the above lemma. Obviously

$$\begin{aligned} \iota_1 : \{\alpha \in C^2(\mathfrak{l},\mathfrak{a}) \mid \alpha(X,Y) = 0,\ \alpha(Z,\mathfrak{l}) \subset \mathfrak{a}^\mathfrak{l}\} &\longrightarrow (\mathfrak{l}/R)^* \otimes \mathfrak{a}^\mathfrak{l} \\ \alpha &\longmapsto \alpha(Z,\cdot) \end{aligned}$$

is an isomorphism. Since $\mathrm{proj}_{\mathfrak{a}^\mathfrak{l}}(d\tau(Z,\cdot)) = -\mathrm{proj}_{\mathfrak{a}^\mathfrak{l}}(\rho(\cdot)\tau(Z)) = 0$ for all $\tau \in C^1(\mathfrak{l},\mathfrak{a})$ the map

$$\begin{aligned} \iota_0 : \quad H^2(\mathfrak{l},\mathfrak{a}) &\longrightarrow (\mathfrak{l}/R)^* \otimes \mathfrak{a}^\mathfrak{l} \\ [\alpha] &\longmapsto \mathrm{proj}_{\mathfrak{a}^\mathfrak{l}}(\alpha(Z,\cdot)) \end{aligned}$$

is well-defined. We have $\iota_1^{-1} \circ \iota_0 \circ \iota^{-1} = \mathrm{Id}$, thus $\iota_0 = \iota_1 \circ \iota$. In particular, ι_0 is an isomorphism.

Lemma 7.6 *If $\mathfrak{a}$ is a Euclidean orthogonal $\mathfrak{l}$-module, then we have*

$$\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp = \{[\alpha,\gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}) \mid [\alpha] \neq 0\} = \iota_Q^{-1}(H^2(\mathfrak{l},\mathfrak{a}) \setminus 0).$$

Proof. All cohomology classes $[0,\gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ are not admissible. Neither (A_0) nor (A_1) is satisfied. For instance, if we assume that (A_0) holds and if we consider $L_0 = Z$, $A_0 = 0$, and $Z_0 \in \mathfrak{l}^*$ defined by $Z_0(X) = Z_0(Y) = 0$, $Z_0(Z) = -\gamma(X,Y,Z)$ we get a contradiction. If $[\alpha] \neq 0$ neither assumption *(i)* of (A_0) nor assumption *(i)* of (A_1) is satisfied. Hence (A_0) and (A_1) hold. Since also (B_0) and (B_1) hold (because $\mathfrak{a}$ is Euclidean) $[\alpha,\gamma]$ is admissible for $[\alpha] \neq 0$. □

We describe automorphisms of $\mathfrak{l}$ by matrices with respect to the basis X, Y, Z of $\mathfrak{l}$. The automorphism group of $\mathfrak{l} = \mathfrak{h}(1)$ equals

$$\mathrm{Aut}(\mathfrak{l}) = \Big\{ S(A,u,x) = \begin{pmatrix} A & 0 \\ x^\top & u \end{pmatrix} \;\Big|\; A \in GL(2,\mathbb{R}),\ \det A = u,\ x \in \mathbb{R}^2 \Big\}.$$

Proposition 7.5 *Let $\mathfrak{g}$ be an indecomposable metric Lie algebra of index 3 such that $\mathfrak{g}/\mathfrak{j}(\mathfrak{g}) \cong \mathfrak{l} := \mathfrak{h}(1)$. Then $\mathfrak{g}$ is isomorphic to one of the indecomposable metric Lie algebras $\mathfrak{d}_{\alpha,0}(\mathfrak{l},\mathfrak{a},\rho)$ with*

(I) $\mathfrak{a} = \mathbb{R}^{2m+1} = \mathbb{R}^{2m} \oplus \mathbb{R}^1$, $m \geq 0$, $\rho = \rho^+_\lambda \oplus \rho_0$, $\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^m$
$\alpha(X,Y) = 0,\ \alpha(X,Z) = A^0_1,\ \alpha(Y,Z) = 0.$

Two such Lie algebras for $\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^m$ and $\bar\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^{\bar m}$ are isomorphic if and only if $m = \bar m$ and either

(a) both $\mathrm{span}\{\lambda(X),\lambda(Y)\}$ *and* $\mathrm{span}\{\bar\lambda(X),\bar\lambda(Y)\}$ *are one-dimensional and*

$$(\mathrm{span}\{\lambda(X),\lambda(Y)\}, \mathbb{R}\cdot\lambda(Y)) = (\mathrm{span}\{\bar\lambda(X),\bar\lambda(Y)\}, \mathbb{R}\cdot\bar\lambda(Y)) \bmod \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m$$

or

(b) both $\mathrm{span}\{\lambda(X),\lambda(Y)\}$ *and* $\mathrm{span}\{\bar\lambda(X),\bar\lambda(Y)\}$ *are two-dimensional and*

$$(\bar\lambda(X) \wedge \bar\lambda(Y), \bar\lambda(Y)) = (r\lambda(X) \wedge \lambda(Y), r^2\lambda(Y)) \bmod \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m$$

for a real number $r \neq 0$.

(II) $\mathfrak{a} = \mathbb{R}^{2m+2} = \mathbb{R}^{2m} \oplus \mathbb{R}^2$, $m \geq 0$, $\rho = \rho^+_\lambda \oplus \rho_0$, $\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^m$,
$\alpha(X,Y) = 0,\ \alpha(X,Z) = A^0_1,\ \alpha(Y,Z) = A^0_2.$
Two such Lie algebras are isomorphic if and only if the $(m \times 2)$-matrices $M_\lambda := (\lambda(X),\lambda(Y))$, $M_{\bar\lambda} := (\bar\lambda(X),\bar\lambda(Y))$ satisfy

$$M_\lambda M_\lambda^\top = M_{\bar\lambda} M_{\bar\lambda}^\top \mod \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m.$$

Proof. We know that $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ is a quadratic extension of $\mathfrak{l}$ by $\mathfrak{a} = \mathbb{R}^n$. We first describe the action of $\mathrm{Aut}(\mathfrak{l})$ on

$$\coprod_{\rho \in \mathrm{Hom}(\mathfrak{l},\mathfrak{so}(n))} \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}_\rho)_\sharp = \coprod_{\rho \in \mathrm{Hom}(\mathfrak{l},\mathfrak{so}(n))} \iota_Q^{-1}(H^2(\mathfrak{l},\mathfrak{a}_\rho) \setminus 0).$$

Suppose $\rho \in \mathrm{Hom}(\mathfrak{l},\mathfrak{so}(n))$, $[\alpha,0] \in \iota_Q^{-1}(H^2(\mathfrak{l},\mathfrak{a}_\rho) \setminus 0)$ and assume $\alpha = \iota([\alpha])$. Then we have for $(S,U) \in G$

$$\iota \circ \iota_Q((S,U)^*[\alpha,0]) \;=\; \iota([U_*^{-1}S^*\alpha]) \;=\; \iota_1^{-1} \circ \iota_0([U_*^{-1}S^*\alpha]) \in \iota(H^2(\mathfrak{l},\mathfrak{a}_{\rho'}) \setminus \{0\}),$$

where $\rho' = (S, U)^* \rho$. Since $\alpha = \iota([\alpha])$ we obtain for $S = S(A, u, x)$

$$\iota \circ \iota_Q(S, U)^*([\alpha, 0]) = \tilde{\alpha} \in H^2(\mathfrak{l}, \mathfrak{a}_{\rho'})$$
$$\tilde{\alpha}(X, Y) = 0, \quad (\tilde{\alpha}(X, Z), \, \tilde{\alpha}(Y, Z)) = u \cdot (U^{-1}\alpha(X, Z), \, U^{-1}\alpha(Y, Z)) \cdot A. \tag{55}$$

Now suppose that $\rho \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{so}(n))$ is given such that $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}_\rho)_\sharp \neq \emptyset$. By Proposition 7.2 we may assume that $\rho = \rho^+_\lambda \oplus \rho_0$ for some $\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^m$, $2m \le n$. Now we consider the G-orbit through an element $[\alpha, 0] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}_\rho)_\sharp$. We assume $\alpha = \iota([\alpha])$, i.e. $\alpha(X, Y) = 0$ and $\alpha(Z, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}_\rho$. Since $\mathfrak{l}$ does not decompose into the direct sum of two non-trivial Lie algebras $[\alpha, 0]$ is indecomposable if and only if $\alpha(Z, \mathfrak{l}) = \mathfrak{a}^{\mathfrak{l}}_\rho$. Hence, we may identify $\mathfrak{a}^{\mathfrak{l}}_\rho$ with $\mathbb{R}^1$ or $\mathbb{R}^2$ spanned by the orthonormal basis A^0_1 or A^0_1, A^0_2, respectively. We may choose a map $A \in GL(2, \mathbb{R})$ such that

$$(\alpha(X, Z), \, \alpha(Y, Z)) \cdot A = \begin{cases} (A^0_1, 0) & \text{if} \quad \dim \alpha(Z, \mathfrak{l}) = 1 \\ (A^0_1, A^0_2) & \text{if} \quad \dim \alpha(Z, \mathfrak{l}) = 2 \end{cases}$$

We set $u := (\det A)^{1/3}$ and $S = S(u^{-1}A, u, 0)$. Then $S^*\rho$ and $S^*\alpha$ satisfy the conditions in (I) if $\dim \alpha(Z, \mathfrak{l}) = 1$ or in (II) if $\dim \alpha(Z, \mathfrak{l}) = 2$.

Now consider two representations $\rho = \rho^+_\lambda \oplus \rho_0$ and $\bar{\rho} = \rho^+_{\bar{\lambda}} \oplus \rho_0$ for $\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^m$ and $\bar{\lambda} \in ((\mathfrak{l}/R)^* \setminus 0)^{\bar{m}}$. Let the 2-form α be defined by $\alpha(X, Y) = 0$, $\alpha(X, Z) = A^0_1$, $\alpha(Y, Z) = 0$. Then $a = [\alpha] \in H^2(\mathfrak{l}, \mathfrak{a}_\rho)$ and $\bar{a} = [\alpha] \in H^2(\mathfrak{l}, \mathfrak{a}_{\bar{\rho}})$. When a and $\bar{a}$ are in the same G-orbit? We have to check under which conditions there is an element $(S, U) \in G$ such that $(S, U)^*\rho = \bar{\rho}$ and $(S, U)^* a = \bar{a}$, which is equivalent to $\iota \circ \iota_Q(S, U)^* \iota_Q^{-1}([\alpha]) = \alpha$. By (55) we can find such an element $(S, U) \in G$ if and only if $m = \bar{m}$ and there are maps $U_0 \in O(1) = \pm 1$ and $A \in GL(2, \mathbb{R})$ such that

$$(\lambda(X), \lambda(Y)) \cdot A = (\bar{\lambda}(X), \bar{\lambda}(Y)) \mod \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m,$$

and

$$\det A \cdot (U_0^{-1} A^0_1, 0) \cdot A = (A^0_1, 0).$$

The last equation is satisfied if and only if $A = \begin{pmatrix} \delta & 0 \\ c & d \end{pmatrix}$ with $\delta, c, d \in \mathbb{R}$ and $U_0(A^0_1) = \delta^2 d \cdot A^0_1$. In particular, $\delta^2 d = \pm 1$. Hence, a and $\bar{a}$ are in the same G-orbit if and only if there are real numbers $c, \delta \in \mathbb{R}$, $\delta \neq 0$, such that

$$(\delta\lambda(X) + c \cdot \lambda(Y), \pm \frac{1}{\delta^2} \cdot \lambda(Y)) = (\bar{\lambda}(X), \bar{\lambda}(Y)) \mod \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m.$$

This proves the isomorphy condition in (I).

For metric Lie algebras of type (II) we proceed in a similar way. Here we obtain that two such Lie algebras for $\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^m$ and $\bar{\lambda} \in ((\mathfrak{l}/R)^* \setminus 0)^{\bar{m}}$ are isomorphic if and only if $m = \bar{m}$ and

$$(\lambda(X), \lambda(Y)) = (\bar{\lambda}(X), \bar{\lambda}(Y)) \mod O(2) \times \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m.$$

Now we use the following general fact, which is true for arbitrary homomorphisms $A, \bar{A} : \mathbb{R}^N \to \mathbb{R}^m$:

$$\exists U \in O(N) : \bar{A} = A \cdot U \quad \Leftrightarrow \quad A\,A^* = \bar{A}\,\bar{A}^* : \mathbb{R}^m \longrightarrow \mathbb{R}^m.$$

Applying this for $\lambda, \bar{\lambda} : \mathfrak{l}/R \to \mathbb{R}^m$, where we identify $\mathfrak{l}/R$ with $\mathbb{R}^2$ using the basis $X + R$, $Y + R$, we obtain the isomorphy condition in (II). $\square$

7.5 The case $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R})$

Proposition 7.6 *Let $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ be an indecomposable metric Lie algebra of index 3 such that $\mathfrak{g}/\mathfrak{j}(\mathfrak{g}) \cong \mathfrak{sl}(2, \mathbb{R})$. Then $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ is isomorphic to exactly one of the indecomposable metric Lie algebras $(\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{sl}(2, \mathbb{R})^*, \langle\cdot\,,\cdot\rangle_c)$, $c \in \mathbb{R}$, where $\langle\cdot\,,\cdot\rangle_c$ is defined by*

$$\langle L_1 + Z_1, L_2 + Z_2\rangle = Z_1(L_2) + Z_2(L_1) + cB_{\mathfrak{l}}(L_1, L_2),$$

for all $L_1, L_2 \in \mathfrak{sl}(2, \mathbb{R})$ and $Z_1, Z_2 \in \mathfrak{sl}(2, \mathbb{R})^*$. Here $B_{\mathfrak{l}}$ denotes the Killing form of $\mathfrak{l} := \mathfrak{sl}(2, \mathbb{R})$.

Proof. Let $\mathfrak{a}$ be an orthogonal $\mathfrak{l}$-module. Since $\mathfrak{l}$ is semi-simple (and, in particular, unimodular) we have $H^2(\mathfrak{l}, \mathfrak{a}) = 0$ and $H^3(\mathfrak{l}) = C^3(\mathfrak{l})$. Therefore

$$\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_\sharp = \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}) = C^3(\mathfrak{l})$$

by Proposition 2.3. In particular, $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0 = C^3(\mathfrak{l})$ if $\mathfrak{a}^{\mathfrak{l}} = 0$ and $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0 = \emptyset$ if $\mathfrak{a}^{\mathfrak{l}} \neq 0$. Since $C^3(\mathfrak{l})$ is one-dimensional any $\gamma \in C^3(\mathfrak{l})$ is a multiple of the non-vanishing 3-form $B_{\mathfrak{l}}([\cdot\,,\cdot], \cdot)$, which is $\mathrm{Aut}(\mathfrak{l})$-invariant. On the other hand each orthogonal representation of $\mathfrak{l}$ on a Euclidean space is trivial. Therefore, by Theorem 5.1, the metric Lie algebra $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ is isomorphic to exactly one of the balanced quadratic extensions $\mathfrak{d}_{0,\gamma}(\mathfrak{l}, 0, 0)$ for $\gamma \in C^3(\mathfrak{l})$. Now it follows from Remark 3.1 that $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ is isomorphic to $\mathfrak{d}'(\mathfrak{l}, cB_{\mathfrak{l}}, 0, 0)$ for a unique $c \in \mathbb{R}$. $\square$

7.6 The case $\mathfrak{l} = \mathfrak{su}(2)$

For $m \in \mathbb{Z}$, $m \geq 0$ we define a set K_m by

$$K_m = \left\{ k = ((k^1, \ldots, k^r), (k_1, \ldots, k_s)) \;\middle|\; \begin{array}{l} k_i, k^i \in \mathbb{N}, \\ 0 < k^1 \leq \ldots \leq k^r,\ 0 < k_1 \leq \ldots \leq k_s, \\ \sum_{i=1}^r (2k^i + 1) + \sum_{i=1}^s 4k_i = m \end{array} \right\}.$$

Here we allow that $r = 0$ or $s = 0$, e.g. $K_0 = \{(\emptyset, \emptyset)\}$.

For $k \in \mathbb{N}$ let $\sigma_k : \mathfrak{su}(2) \to \mathfrak{so}(2k+1)$ and $\sigma'_k : \mathfrak{su}(2) \to \mathfrak{so}(4k)$ be the non-trivial irreducible real representations of $\mathfrak{su}(2)$. For $k = ((k^1, \ldots, k^r), (k_1, \ldots, k_s)) \in K_m$ let the representation ρ_k of $\mathfrak{su}(2)$ on $\mathbb{R}^m$ be the direct sum

$$\rho_k = \bigoplus_{i=1}^r \sigma_{k^i} \oplus \bigoplus_{i=1}^s \sigma'_{k_i}.$$

In particular, if $k = (\emptyset, \emptyset) \in K_0$, then ρ_k is the zero representation.

Let $B_{\mathfrak{l}}$ denote the Killing form of $\mathfrak{l} = \mathfrak{su}(2)$. In the following proposition we will use the modified quadratic extensions defined in Remark 3.1.

Proposition 7.7 *If* $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ *is an indecomposable metric Lie algebra of index 3 such that* $\mathfrak{g}/\mathfrak{j}(\mathfrak{g}) \cong \mathfrak{l} = \mathfrak{su}(2)$*, then* $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ *is isomorphic to exactly one of the metric Lie algebras* $\mathfrak{d}'_{0,0}(\mathfrak{l}, cB_{\mathfrak{l}}, \mathbb{R}^m, \rho_k)$*, where* $m > 0$*,* $k \in K_m$ *and* $c \in \mathbb{R}$*.*

In particular, if $\dim \mathfrak{g} = 6$*, then* $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ *is isomorphic to exactly one of the metric Lie algebras* $\mathfrak{d}'_{0,0}(\mathfrak{l}, cB_{\mathfrak{l}}, 0, 0) = \mathfrak{su}(2) \ltimes \mathfrak{su}(2)^*$ *for* $c \in \mathbb{R}$*.*

Proof. As in the case of $\mathfrak{sl}(2,\mathbb{R})$ we have $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 = C^3(\mathfrak{l})$ if $\mathfrak{a}^{\mathfrak{l}} = 0$ and $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 = \emptyset$ if $\mathfrak{a}^{\mathfrak{l}} \neq 0$. Each orthogonal representation $(\rho, \mathfrak{a})$ of $\mathfrak{l}$ with $\mathfrak{a}^{\mathfrak{l}} = 0$ is equivalent to a unique representation ρ_k, $k \in K_m$, $m \geq 0$.

Furthermore, we have $C^3(\mathfrak{l}) = \mathbb{R} \cdot \gamma_0$ for $\gamma_0 := B_{\mathfrak{l}}([\cdot\,,\cdot],\cdot)$. As in the case of $\mathfrak{sl}(2,\mathbb{R})$ we can now use Theorem 5.1 and Remark 3.1 to finish the proof. □

7.7 The case $\mathfrak{l} = \mathbb{R}^k$, $k = 1, 2$

If $\mathfrak{l} = \mathbb{R}^k$, $k = 1, 2$, then we can identify

$$\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}) = H^2(\mathfrak{l},\mathfrak{a}) = C^2(\mathfrak{l},\mathfrak{a}^{\mathfrak{l}}).$$

Lemma 7.7 *For* $\mathfrak{l} = \mathbb{R}^k$*,* $k = 1, 2$ *we have*

$$\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 = \{\alpha \in C^2(\mathfrak{l},\mathfrak{a}^{\mathfrak{l}}) \mid \alpha(\mathfrak{l},\mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}} \text{ non-degenerate, } \alpha \text{ indecomposable}\}.$$

Proof. Condition (B_0) implies that $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$ is contained in the set on the r. h. s. Now let $\alpha \in C^2(\mathfrak{l},\mathfrak{a}^{\mathfrak{l}})$ be indecomposable and such that $\alpha(\mathfrak{l},\mathfrak{l})$ is non-degenerate. Assume that α is not admissible. Then Condition (A_0) cannot be satisfied. Hence there are elements $L_0 \in \ker\rho$, $L_0 \neq 0$ and $A_0 \in \mathfrak{a}$ such that:

$$\alpha(\cdot, L_0) = \rho(\cdot)(A_0). \tag{56}$$

Since $\alpha(\mathfrak{l}, L_0) \subset \mathfrak{a}^{\mathfrak{l}}$ Equation (56) implies $\alpha(L_0,\cdot) = 0$. Since on the other hand $L_0 \in \ker\rho$ and $L_0 \neq 0$ this is a contradiction to the indecomposability of α. □

A pair $(\mathfrak{l},\mathfrak{a})$ is called decomposable if it is a non-trivial direct sum of two pairs. Otherwise the pair is called indecomposable.

Corollary 7.1 *For* $\mathfrak{l} = \mathbb{R}^k$*,* $k = 1, 2$ *we have*

$$\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 = \begin{cases} \emptyset & \textit{if} \quad \dim \mathfrak{a}^{\mathfrak{l}} > 1 \\ C^2(\mathfrak{l},\mathfrak{a}^{\mathfrak{l}}) \setminus \{0\} & \textit{if} \quad \dim \mathfrak{a}^{\mathfrak{l}} = 1 \\ \{0\} & \textit{if} \quad \dim \mathfrak{a}^{\mathfrak{l}} = 0,\ (\mathfrak{l},\mathfrak{a}) \textit{ is indecomposable} \\ \emptyset & \textit{if} \quad \dim \mathfrak{a}^{\mathfrak{l}} = 0,\ (\mathfrak{l},\mathfrak{a}) \textit{ is decomposable} \end{cases}.$$

First we consider the case $\mathfrak{l} = \mathbb{R}^2$. We fix a basis $\{Y, Z\}$ of $\mathbb{R}^2$.

Proposition 7.8 *If $\mathfrak{g}$ is an indecomposable metric Lie algebra of index 3 such that $\mathfrak{g}/\mathfrak{j}(\mathfrak{g}) \cong \mathfrak{l} := \mathbb{R}^2$, then $\mathfrak{g}$ is isomorphic to one of the following indecomposable Lie algebras $\mathfrak{d}_{\alpha,0}(\mathfrak{l},\mathfrak{a},\rho)$ with*

(I) $\mathfrak{a} = \mathbb{R}^{1,2m} = \mathbb{R}^{2m} \oplus \mathbb{R}^{1,0}$, $m \geq 0$, $\rho = \rho_\lambda^+ \oplus \rho_0$, *where* $\lambda \in (\mathfrak{l}^* \setminus 0)^m$,
$\alpha(Y,Z) = A_1^0$.
Two such Lie algebras for $\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^m$ *and* $\bar\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^{\bar m}$ *are isomorphic if and only if* $m = \bar m$ *and*

$$(\,\mathrm{span}\{\lambda(Y),\lambda(Z)\}, \lambda(Y)\wedge\lambda(Z)\,) = (\,\mathrm{span}\{\bar\lambda(Y),\bar\lambda(Z)\}, \pm\bar\lambda(Y)\wedge\bar\lambda(Z)\,) \mod \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m.$$

(II) $\mathfrak{a} = \mathbb{R}^{1,2m+2} = \mathbb{R}^{2m} \oplus \mathbb{R}^{1,1} \oplus \mathbb{R}^1$, $m \geq 0$, $\rho = \rho_\lambda^+ \oplus \rho'_\mu \oplus \rho_0$,
where $\lambda \in (\mathfrak{l}^* \setminus 0)^m$, $\mu \in \mathfrak{l}^*$ *such that*
$\mu(Y) = 1$, $\mu(Z) = 0$,
$\alpha(Y,Z) = A_1^0$.
Two such Lie algebras for $\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^m$ *and* $\bar\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^{\bar m}$ *are isomorphic if and only if* $m = \bar m$ *and either*

(a) $\mathrm{span}\{\lambda(Y),\lambda(Z)\}$ *and* $\mathrm{span}\{\bar\lambda(Y),\bar\lambda(Z)\}$ *are one-dimensional, and*

$$\lambda(Z) \neq 0 \text{ and } \lambda(Z) = \bar\lambda(Z) \mod \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m$$

or

$$\lambda(Z) = \bar\lambda(Z) = 0 \text{ and } \lambda(Y) = \bar\lambda(Y) \mod \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m$$

or

(b) $\mathrm{span}\{\lambda(Y),\lambda(Z)\}$ *and* $\mathrm{span}\{\bar\lambda(Y),\bar\lambda(Z)\}$ *are two-dimensional and*

$$(\lambda(Y)\wedge\lambda(Z)\,,\lambda(Z))) = (\pm\bar\lambda(Y)\wedge\bar\lambda(Z)\,,\lambda(Z)) \mod \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m.$$

(III) $\mathfrak{a} = \mathbb{R}^{1,2m+1} = \mathbb{R}^{2m} \oplus \mathbb{R}^{1,1}$, $m \geq 2$, $\rho = \rho_\lambda^+ \oplus \rho'_\mu$,
where $\lambda \in (\mathfrak{l}^* \setminus 0)^m$ *is such that the set* $\{(1,0)\} \cup \{(\lambda^i(Y),\lambda^i(Z)) \mid i = 1,\dots,m\}$ *is not contained in the union of two 1-dimensional subspaces of* $\mathbb{R}^2$,
$\mu \in \mathfrak{l}^*$ *is given by* $\mu(Y) = 1$, $\mu(Z) = 0$,
$\alpha = 0$.
Two such Lie algebras for $\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^m$ *and* $\bar\lambda \in ((\mathfrak{l}/R)^* \setminus 0)^{\bar m}$ *are isomorphic if and only if* $m = \bar m$ *and*

$$\lambda(Y) + \mathbb{R}\cdot\lambda(Z) = \bar\lambda(Y) + \mathbb{R}\cdot\bar\lambda(Z) \mod \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m.$$

Proof. Let $(\rho,\mathfrak{a})$ be such that $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 \neq \emptyset$. By Corollary 7.1 either $\dim \mathfrak{a}^\mathfrak{l} = 1$ or $\dim \mathfrak{a}^\mathfrak{l} = 0$. If $\dim \mathfrak{a}^\mathfrak{l} = 1$, then either $\mathfrak{a}^\mathfrak{l} = \mathbb{R}^{1,0}$ or $\mathfrak{a}^\mathfrak{l} = \mathbb{R}^1$.

Let us first consider the case $\mathfrak{a}^\mathfrak{l} = \mathbb{R}^{1,0}$. This will lead to Lie algebras of type (I) in the proposition. By Proposition 7.2 we may assume $\mathfrak{a} = \mathbb{R}^{1,2m}$ and $\rho = \rho_\lambda^+ \oplus \rho_0$, $\lambda \in (\mathfrak{l}^* \setminus 0)^m$. Suppose $\alpha \in C^2(\mathfrak{l},\mathfrak{a}^\mathfrak{l}) \setminus \{0\}$ $(= \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0)$. Let A_1^0 be a fixed unit vector in $\mathfrak{a}^\mathfrak{l} = \mathbb{R}^{1,0}$. It is easy to find a map $S \in \mathrm{Aut}(\mathfrak{l}) = GL(2,\mathbb{R})$ such that $S^*\alpha(Y,Z) = A_1^0$.

Then $S^*\rho$, $S^*\alpha$ satisfy the conditions in (I). Now let $\rho = \rho_\lambda^+ \oplus \rho_0$ and $\bar\rho = \rho_{\bar\lambda}^+ \oplus \rho_0$ be representations on $\mathbb{R}^{2m} \oplus \mathbb{R}^{1,0}$ for different $\lambda, \bar\lambda \in (\mathfrak{l}^* \setminus 0)^m$ and let α be defined by $\alpha(Y,Z) = A_1^0$. Then α defines cohomology classes $a \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}_\rho)$ and $\bar a \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}_{\bar\rho})$. We have to check under which conditions $a = \bar a \mod G$ holds. First we note that $(S,U)^*\rho = \bar\rho$ implies $U(\mathfrak{a}^\mathfrak{l}) = \mathfrak{a}^\mathfrak{l}$ (thus $U|_{\mathfrak{a}^\mathfrak{l}} = \pm 1$) and $(S, U|_{(\mathfrak{a}^\mathfrak{l})^\perp})^*\rho_\lambda = \rho_{\bar\lambda}$. Using this it is easy to see that $a = \bar a \mod G$ if and only if there is a map $S \in GL(2,\mathbb{R})$ such that $\alpha(SY, SZ) = \pm\alpha(Y,Z)$ and $[S^*\lambda] = [\bar\lambda] \in \Lambda_m$. This is the case if and only if there is a map $S \in SL^\pm(2,\mathbb{R}) := \{S \in GL(2,\mathbb{R}) \mid \det S = \pm 1\}$ satisfying $[S^*\lambda] = [\bar\lambda] \in \Lambda_m$, which is equivalent to the condition in (I).

Now let us suppose $\mathfrak{a}^\mathfrak{l} = \mathbb{R}^1$. This will lead to Lie algebras of type (II). Here we may assume $\mathfrak{a} = \mathbb{R}^{1,2m+2}$ and $\rho = \rho_\lambda^+ \oplus \rho'_\mu \oplus \rho_0$, $\lambda \in (\mathfrak{l}^* \setminus 0)^m$, $\mu \in \mathfrak{l}^* \setminus 0$. If $\alpha \in C^2(\mathfrak{l}, \mathfrak{a}^\mathfrak{l}) \setminus 0 (= \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0)$ and A_1^0 is a fixed unit vector in $\mathfrak{a}^\mathfrak{l} = \mathbb{R}^1$, then one can easily find a map $S \in GL(2,\mathbb{R})$ such that $S^*\alpha(Y,Z) = A_1^0$, $S^*\mu(Y) = 1$ and $S^*\mu(Z) = 0$. Then $S^*\rho$, $S^*\alpha$ satisfy the conditions in (II). Now let α and μ be as in (II) and consider $\rho = \rho_\lambda^+ \oplus \rho'_\mu \oplus \rho_0$ and $\bar\rho = \rho_{\bar\lambda}^+ \oplus \rho'_\mu \oplus \rho_0$ for $\lambda, \bar\lambda \in (\mathfrak{l}^* \setminus 0)^m$. One proves in a similar way as above that $a = [\alpha] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}_\rho)$ and $\bar a = [\alpha] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}_{\bar\rho})$ are in the same G-orbit if and only if there is a map $S \in GL(2,\mathbb{R})$ such that $\alpha(SY, SZ) = \pm\alpha(Y,Z)$, $S^*\mu = \pm\mu$ and $[S^*\lambda] = [\bar\lambda] \in \Lambda_m$. This is the case if and only if there exists a $c \in \mathbb{R}$ such that

$$(\pm\lambda(Y) + c\lambda(Z), \lambda(Z))) = (\bar\lambda(Y), \bar\lambda(Z)) \mod \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m.$$

This yields the isomorphism condition in *(II)*.

Finally we consider the case $\mathfrak{a}^\mathfrak{l} = 0$. Here we have $\alpha = 0$ and we may assume $\mathfrak{a} = \mathbb{R}^{1,2m}$ and $\rho = \rho_\lambda^+ \oplus \rho'_\mu$, $\lambda \in (\mathfrak{l}^* \setminus 0)^m$, $\mu \in \mathfrak{l}^* \setminus 0$.

The indecomposability of $\mathfrak{g}$ is equivalent to the line condition on λ.

We have $S^*\mu = \pm\mu$ and $[S^*\lambda] = [\bar\lambda] \in \Lambda_m$ if and only if there exists $c, d \in \mathbb{R}$ such that $d \neq 0$ and

$$(\pm\lambda(Y) + c\lambda(Z), d\lambda(Z))) = (\bar\lambda(Y), \bar\lambda(Z)) \mod \mathfrak{S}_m \ltimes (\mathbb{Z}_2)^m.$$

□

If $\mathfrak{l} = \mathbb{R} = \mathbb{R} \cdot X$, then it is easy to prove the following classification result. We identify $\lambda \in (\mathfrak{l}_0^*)^m$ with $\lambda(X) \in \mathbb{R}^m$ and $\mu \in (\mathfrak{l}_0^*)^r$ with $\mu(X) \in \mathbb{R}^r$.

Proposition 7.9 *If $\mathfrak{g}$ is an indecomposable metric Lie algebra of index 3 such that $\mathfrak{g}/\mathfrak{i}(\mathfrak{g})^\perp \cong \mathfrak{l} := \mathbb{R}^1$, then $\mathfrak{g}$ is isomorphic to exactly one of the following indecomposable Lie algebras $\mathfrak{d}_{0,0}(\mathfrak{l}, \mathfrak{a}, \rho)$ with*

(I) $\mathfrak{a} = \mathbb{R}^{2,2m+2} = \mathbb{R}^{2m} \oplus \mathbb{R}^{2,2}$, $m \geq 0$, $\rho = \rho_\lambda^+ \oplus \rho''_{(\mu,\nu)}$,
where $\lambda \in \mathbb{R}^m$, $0 < \lambda^1 \leq \lambda^2 \leq \ldots \leq \lambda^m$, $\mu = 1$, $\nu \in \mathbb{R}$, $\nu \neq 0$*;*

(II) $\mathfrak{a} = \mathbb{R}^{2,2m} = \mathbb{R}^{2m} \oplus \mathbb{R}^{2,0}$, $m \geq 0$, $\rho = \rho_\lambda^+ \oplus \rho_{\lambda^0}^-$,
where $\lambda^0 = 1$, $\lambda = (\lambda^1, \ldots, \lambda^m) \in \mathbb{R}^m$, $0 < \lambda^1 \leq \lambda^2 \leq \ldots \leq \lambda^m$*;*

(III) $\mathfrak{a} = \mathbb{R}^{2,2m+2} = \mathbb{R}^{2m} \oplus \mathbb{R}^{2,2}$, $m \geq 0$, $\rho = \rho^+_\lambda \oplus \rho'_\mu$,
where $\lambda = (\lambda^1, \dots, \lambda^m) \in \mathbb{R}^m$, $0 < \lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^m$,
$\mu = (\mu^1, \mu^2) \in \mathbb{R}^2$, $1 = \mu^1 \leq \mu^2$.

7.8 Summary

Propositions 7.1 and 7.3 – 7.8 yield a classification of indecomposable non-simple metric Lie algebras of index 3 up to the case where $\mathfrak{g}/\mathfrak{j}(\mathfrak{g})$ is isomorphic to $\mathbb{R}^3$. In this case $\mathfrak{a}$ is Euclidean and $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ is an indecomposable metric Lie algebra of index 3 with maximal isotropic centre. For a classification of these algebras see [KO 02], Theorem 5.1.

It remains to determine all simple metric Lie algebras of index 3. This is done by checking the list of all simple Lie algebras.

Finally we obtain the following classification result for metric Lie algebras of index 3.

Theorem 7.1 *If* $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ *is a simple metric Lie algebra of index 3, then* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{su}(2)$ *or* $\mathfrak{sl}(3,\mathbb{R})$ *and* $\langle\cdot\,,\cdot\rangle$ *is a positive multiple of the Killing form or it is isomorphic to* $\mathfrak{sl}(2,\mathbb{C})$ *and* $\langle\cdot\,,\cdot\rangle$ *is a non-zero multiple of the Killing form.*

If $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ *is a non-simple indecomposable metric Lie algebra of index 3, then* $(\mathfrak{g}, \langle\cdot\,,\cdot\rangle)$ *is isomorphic to exactly one Lie algebra* $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}, \rho)$ *with the following data:*

$\mathfrak{l}$	$\mathfrak{a}$	ρ	α (characteristic property)	γ	parameters	detailed description in
$\mathfrak{n}(2)$	$\mathbb{R}^{2m}$	ρ_λ^+	$\alpha = 0$	$\gamma_\kappa \neq 0$	$m \geq 0,\ \kappa = \pm 1,\ [\lambda] \in \Lambda_m$	Prop. 7.3 (Ia, b)
	$\mathbb{R}^{2m} \oplus \mathbb{R}^1$	$\rho_\lambda^+ \oplus \rho_0$	$\alpha(\mathfrak{l}, \mathfrak{l}) = \mathbb{R}^1$	$\gamma = 0$	$m \geq 0,\ [\lambda] \in \Lambda_m$	Prop. 7.3 *(II)*
	$\mathbb{R}^{2m+2} \oplus \mathbb{R}^1$	$\rho_{\lambda'}^+ \oplus \rho_0$	$\alpha_r(\mathfrak{l}, \mathfrak{l}) = \mathbb{R}^2 \oplus \mathbb{R}^1$	$\gamma = 0$	$m \geq 0,\ \lambda' = (\lambda, 1),\ [\lambda] \in \Lambda_m, r \in \mathbb{R}, r > 0$	Prop. 7.3 *(III)*
	$\mathbb{R}^{2m+2}$	$\rho_{\lambda'}^+$	$\alpha(\mathfrak{l}, \mathfrak{l}) = \mathbb{R}^2$	$\gamma = 0$	$m \geq 0,\ \lambda' = (\lambda, 1),\ [\lambda] \in \Lambda_m$	Prop. 7.3 *(IV)*
$\mathfrak{r}_{3,-1}$	$\mathbb{R}^{2m} \oplus \mathbb{R}^1$	$\rho_\lambda^+ \oplus \rho_0$	$\alpha(\mathfrak{l}, \mathfrak{l}) = \mathbb{R}^1$	$\gamma = 0$	$m \geq 0,\ [\lambda] \in \Lambda_m$	Prop. 7.4 *(I)*
	$\mathbb{R}^{2m}$	ρ_λ^+	$\alpha = 0$	$\gamma \neq 0$	$m \geq 0,\ [\lambda] \in \Lambda_m$	Prop. 7.4 *(II)*
$\mathfrak{h}(1)$	$\mathbb{R}^{2m} \oplus \mathbb{R}^1$	$\rho_\lambda^+ \oplus \rho_0$	$\alpha(\mathfrak{l}, \mathfrak{l}) = \mathbb{R}^1$	$\gamma = 0$	$m \geq 0,\ [\lambda] \in \Lambda_m/(\mathbb{R}^* \ltimes \mathbb{R})$	Prop. 7.5 *(I)*
	$\mathbb{R}^{2m} \oplus \mathbb{R}^2$	$\rho_\lambda^+ \oplus \rho_0$	$\alpha(\mathfrak{l}, \mathfrak{l}) = \mathbb{R}^2$	$\gamma = 0$	$m \geq 0,\ [\lambda] \in \Lambda_m/O(2)$	Prop. 7.5 *(II)*
$\mathfrak{sl}(2, \mathbb{R})$	0	–	$\alpha = 0$	γ_c	$c \in \mathbb{R}$	Prop. 7.6
$\mathfrak{su}(2)$	$\mathbb{R}^m$	ρ_k	$\alpha = 0$	γ_c	$m \geq 0,\ k \in K_m,\ c \in \mathbb{R}$	Prop. 7.7
$\mathbb{R}^1$	$\mathbb{R}^{2m} \oplus \mathbb{R}^{2,2}$	$\rho_\lambda^+ \oplus \rho''_{(1,\nu)}$	$\alpha = 0$	$\gamma = 0$	$m \geq 0,\ [\lambda] \in \Lambda_m,\ \nu \in \mathbb{R} \setminus \{0\}$	Prop. 7.9 *(I)*
	$\mathbb{R}^{2m} \oplus \mathbb{R}^{2,0}$	$\rho_\lambda^+ \oplus \rho_1^-$	$\alpha = 0$	$\gamma = 0$	$m \geq 0,\ [\lambda] \in \Lambda_m$	Prop. 7.9 *(II)*
	$\mathbb{R}^{2m} \oplus \mathbb{R}^{2,2}$	$\rho_\lambda^+ \oplus \rho'_{(1,\mu)}$	$\alpha = 0$	$\gamma = 0$	$m \geq 0,\ [\lambda] \in \Lambda_m,\ \mu \in \mathbb{R},\ \mu \geq 1$	Prop. 7.9 *(III)*
$\mathbb{R}^2$	$\mathbb{R}^{2m} \oplus \mathbb{R}^{1,0}$	$\rho_\lambda^+ \oplus \rho_0$	$\alpha(\mathfrak{l}, \mathfrak{l}) = \mathbb{R}^{1,0}$	$\gamma = 0$	$m \geq 0,\ [\lambda] \in \Lambda_m/SL^{\pm}(2, \mathbb{R})$	Prop. 7.8 *(I)*
	$\mathbb{R}^{2m} \oplus \mathbb{R}^{1,1} \oplus \mathbb{R}^1$	$\rho_\lambda^+ \oplus \rho'_\mu \oplus \rho_0$	$\alpha(\mathfrak{l}, \mathfrak{l}) = \mathbb{R}^1$	$\gamma = 0$	$m \geq 0,\ [\lambda] \in \Lambda_m/(\mathbb{Z}_2 \ltimes \mathbb{R})$	Prop. 7.8 *(II)*
	$\mathbb{R}^{2m} \oplus \mathbb{R}^{1,1}$	$\rho_\lambda^+ \oplus \rho'_\mu$	$\alpha = 0$	$\gamma = 0$	$m \geq 2,\ [\lambda] \in \mathcal{O},\ \mathcal{O} \subset \Lambda_m/(\mathbb{R}^* \ltimes \mathbb{R})$ open	Prop. 7.8 *(III)*
$\mathbb{R}^3$	$\mathbb{R}^{2m}$	ρ_λ^+	$\alpha = 0$	$\gamma = 0$	$m \geq 4,\ [\lambda] \in \mathcal{O}_1,\ \mathcal{O}_1 \subset \Lambda_m/GL(3, \mathbb{R})$ open	[KO 02],
	$\mathbb{R}^{2m}$	ρ_λ^+	$\alpha = 0$	$\gamma \neq 0$	$m \geq 0,\ [\lambda] \in \Lambda_m/SL(3, \mathbb{R})$	Theorem 5.3
	$\mathbb{R}^{2m} \oplus \mathbb{R}^1$	$\rho_\lambda^+ \oplus \rho_0$	$\alpha(\mathfrak{l}, \mathfrak{l}) = \mathbb{R}^1$	$\gamma = 0$	$m \geq 2,\ [\lambda] \in \mathcal{O}_2,\ \mathcal{O}_2 \subset \Lambda_m/((SL^{\pm}(2, \mathbb{R}) \times \mathbb{R}^*) \ltimes \mathbb{R}^2)$ open	
	$\mathbb{R}^{2m} \oplus \mathbb{R}^2$	$\rho_\lambda^+ \oplus \rho_0$	$\alpha(\mathfrak{l}, \mathfrak{l}) = \mathbb{R}^2$	$\gamma = 0$	$m \geq 0,\ [\lambda] \in \Lambda_m/(CO(2) \ltimes \mathbb{R}^2)$	
	$\mathbb{R}^{2m} \oplus \mathbb{R}^3$	$\rho_\lambda^+ \oplus \rho_0$	$\alpha(\mathfrak{l}, \mathfrak{l}) = \mathbb{R}^3$	$\gamma = 0$	$m \geq 0,\ [\lambda] \in \Lambda_m/O(3)$	

With some effort, it should be possible to classify metric Lie algebras of index $4, 5, \ldots$ in the same way. But very soon the method will reach its limits. On the one hand, there is the problem of classification of admissible Lie algebras. On the other hand, also the explicit determination of the orbit spaces $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}_\rho)_0/G_\rho$ sometimes leads to unsolved classification problems in multilinear algebra (an example is discussed in [KO 02], Remark 5.3).

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